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The density of rational points on a certain singular cubic surface

T.D. Browning

School of Mathematics, Bristol University, Bristol BS8 1TW, UK

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Abstract

We show that the number of nontrivial rational points of height at most B, which lie on the cubic surface $x_1x_2x_3 = x_4(x_1 + x_2 + x_3)^2$, has order of magnitude $B(\log B)^6$. This agrees with Manin's conjecture.

1. Introduction

The purpose of this paper is to investigate the distribution of rational points on the singular cubic surface $X \subset \mathbb{P}^3$, given by the equation

$$x_1x_2x_3 = x_4(x_1 + x_2 + x_3)^2.$$

This surface has a unique singular point $[0, 0, 0, 1]$ which is of type D_4, and contains precisely 6 lines [2, Lemma 4]. These lines are all defined over $\mathbb{Q}$ and are given by

$$x_i = x_4 = 0, \qquad x_i = x_j + x_k = 0,$$

for $\{i, j, k\} = \{1, 2, 3\}$. We shall denote by $U \subset X$ the open subset formed by deleting the lines from X.

E-mail address: t.d.browning@bristol.ac.uk.

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Given a rational point $x=[x_1,x_2,x_3,x_4]\in\mathbb{P}^3(\mathbb{Q})$ such that x_1,x_2,x_3,x_4 are relatively prime integers, let $H(x)=\max_{1\leqslant i\leqslant 4}|x_i|$ denote its anticanonical height, metrized by the choice of norm $\max_{1\leqslant i\leqslant 4}|x_i|$. Then for any $B\geqslant 1$, we shall be concerned with estimating the quantity

$$N_{U,H}(B)=\#\{x\in U\cap\mathbb{P}^3(\mathbb{Q}):\ H(x)\leqslant B\}.$$

Manin [4] has provided a very general conjecture concerning the distribution of rational points on Fano varieties. In our case it predicts that there exists a positive constant $c_{X,H}$ such that

$$N_{U,H}(B)\sim c_{X,H}B(\log B)^6,$$

as $B\to\infty$. Here the exponent of $\log B$ is one less than the rank of the Picard group $\operatorname{Pic}\tilde{X}$ of $\tilde{X}$, where $\tilde{X}$ denotes the minimal desingularisation of X. In fact this sort of asymptotic formula is conjectured to hold for any cubic surface with canonical singular locus. Although there has been increasing interest in Manin's conjecture for cubic surfaces, it has only been completely settled in particularly simple cases such as the toric variety

$$x_1x_2x_3=x_4^3.$$

This can be found in the work of de la Bretèche [1], for example.

More recently, Heath-Brown [10] has established upper and lower bounds for the density of nontrivial rational points on the Cayley cubic surface

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}=0,$$

which agree with Manin's conjecture. This is a cubic surface containing four A_1 singularities, which is the maximal number of singularities that a nonruled cubic surface can have. The principal tool in Heath-Brown's work is a passage to the universal torsor above the minimal desingularisation of the Cayley cubic. Originally introduced by Colliot-Thélène and Sansuc to aid in the study of the Hasse principle and weak approximation, universal torsors were first used by Peyre [12] and Salberger [13] in the context of counting rational points of bounded height. After establishing a bijection between the rational points on the Cayley cubic and the integer points on the universal torsor, which in this setting is given explicitly by nine equations in thirteen variables, Heath-Brown proceeds by applying methods from the geometry of numbers to count integer solutions to certain ternary linear equations.

Our present work is largely inspired by Heath-Brown's treatment of the Cayley cubic surface. We are now in a position to state our main result.

Theorem. *We have*

$$B(\log B)^6\ll N_{U,H}(B)\ll B(\log B)^6. \tag{1.1}$$

Of the two bounds in our theorem, the lower bound $N_{U,H}(B)\gg B(\log B)^6$ is routine. It will follow from relatively minor adjustments to Heath-Brown's treatment of the Cayley cubic. Establishing the upper bound in (1.1), however, is by far the most challenging component of this paper and the proof has two fundamental ingredients. The first is a translation of the problem to the universal torsor above $\tilde{X}$, which in this setting has the affine embedding

$$s_0s_1s_2s_3u_1u_2u_3 = y_1u_1s_1^2 + y_2u_2s_2^2 + y_3u_3s_3^2. \tag{1.2}$$

This has been calculated by Hassett and Tschinkel [6, Section 4], although we shall present our own deduction of this equation in Section 4 below. The universal torsor can be thought of as serving to encode factorisation information about the integer solutions to the original equation. In practical terms, it allows us to work with a larger number of variables, all of which are smaller in modulus than the original variables x_1, x_2, x_3, x_4. The second main ingredient in our proof of the upper bound involves studying the distribution of integer solutions to the equation obtained by setting $s_0 = 0$ in (1.2). This is the focus of Section 3 and relies upon lattice methods to count integer solutions to ternary linear and quadratic forms. It seems worthwhile highlighting the fact that this need to consider the contribution from quadratic equations marks a significant departure from Heath-Brown's treatment of the Cayley cubic. In particular, we shall need to pay careful attention to the fact that almost all ternary quadratic forms do not have an integer solution.

Our work draws upon a diverse range of techniques. In addition to the geometry of numbers used to study linear and quadratic forms in Section 2.1, we make use of the large sieve inequality and real character sum estimates in Section 2.2.

2. Preliminary estimates

We begin by introducing some conventions regarding our choice of notation. Throughout this paper the letters i, j, k will denote generic distinct indices from the set $\{1, 2, 3\}$. We shall use $\mathbb{N}$ to denote the set of positive integers, and for any $n \geqslant 2$ it will be convenient to let Z^n denote the set of primitive vectors in $\mathbb{Z}^n$, where $\mathbf{v} \in \mathbb{Z}^n$ is said to be primitive if $\gcd(v_1, \dots, v_n) = 1$. Similarly, we let N^n denote the set of primitive vectors in $\mathbb{N}^n$. Furthermore, we let Z_*^n denote the subset of $\mathbf{v} \in Z^n$ for which $v_1 \cdots v_n \neq 0$. Upon writing

$$F(\mathbf{x}) = x_1x_2x_3 - x_4(x_1 + x_2 + x_3)^2,$$

it therefore follows that

$$N_{U,H}(B) = \frac{1}{2}\#\Big\{\mathbf{x} \in Z_*^4\colon \max_{1 \leqslant i \leqslant 4} |x_i| \leqslant B,\ F(\mathbf{x}) = 0\Big\}, \tag{2.1}$$

since $\mathbf{x}$ and $-\mathbf{x}$ represent the same point in $\mathbb{P}^3$. It will be convenient to collect together some technical results that will be useful to us.

2.1. *The geometry of numbers and ternary forms*

Several of our arguments will involve estimating the number of primitive integer solutions to certain ternary homogeneous polynomial equations, which lie in lopsided regions. In the case of linear equations, such an estimate is provided by the following result of Heath-Brown [7, Lemma 3].

Lemma 1. *Let* $\mathbf{h} \in Z^3$ *and let* $W_i > 0$*. Then the number of* $\mathbf{w} \in Z^3$ *for which* $\sum_{i=1}^{3} h_i w_i = 0$*, and* $|w_i| \leqslant W_i$*, is*

$$\leqslant 4 + 12\pi \frac{W_1 W_2 W_3}{\max |h_i| W_i}.$$

We shall also need a result which handles the corresponding problem for diagonal quadratic equations. For this we turn to the following result, in which $\omega(n)$ denotes the number of distinct prime factors of $n \in \mathbb{N}$.

Lemma 2. *Let* $\mathbf{g}, \mathbf{h} \in Z_*^3$*, with* $g_1 g_2 g_3$ *square-free, and let* $W_i > 0$*. Then the number of* $\mathbf{w} \in Z^3$ *for which* $\sum_{i=1}^{3} g_i h_i w_i^2 = 0$*, and* $|w_i| \leqslant W_i$*, is*

$$\ll \left(1 + \sqrt{\frac{W_1 W_2 W_3 D_{\mathbf{g},\mathbf{h}}^{3/2}}{|h_1 h_2 h_3|}}\right) 2^{\omega(h_1 h_2 h_3)},$$

where $D_{\mathbf{g},\mathbf{h}}$ *is the product of greatest common divisors*

$$\gcd(h_1 h_2, h_1 h_3, h_2 h_3) \gcd(g_1, h_2 h_3) \gcd(g_2, h_1 h_3) \gcd(g_3, h_1 h_2).$$

Lemma 2 will follow from a rather straightforward modification to the proof of Heath-Brown [9, Theorem 2]. In fact, Heath-Brown establishes a version of Lemma 2 with $\mathbf{g} = (1, 1, 1)$ and $d_3(|h_1 h_2 h_3|)$ in place of $2^{\omega(h_1 h_2 h_3)}$, where $d_k(n)$ denotes the number of representations of n as a product of k positive integers, for any $k, n \in \mathbb{N}$. It is perhaps worth pointing out that whereas $d_3(n)$ has average order $\frac{1}{2}(\log n)^2$, the function $2^{\omega(n)}$ has average order $\zeta(2)^{-1} \log n$. This saving plays an important role in our work.

In order to prove Lemma 2 we recall that the original idea behind the proof of [9, Theorem 2] is to view the equation $\sum_{i=1}^{3} g_i h_i w_i^2 = 0$ as a collection of lattice conditions upon the solutions $\mathbf{w} \in \mathbb{Z}^3$. Let p be any prime divisor of $h_1 h_2 h_3$, and assume without loss of generality that

$$0 \leqslant \nu_p(h_1) \leqslant \nu_p(h_2) \leqslant \nu_p(h_3),$$

where $\nu_p(n)$ denotes the p-adic order of any $n \in \mathbb{N}$. In particular, it follows that $\nu_p(h_1) = 0$, since $\mathbf{h}$ is primitive. We shall only consider here the case in which p is an odd prime. The case $p = 2$ is handled along similar lines. Since $g_1 g_2 g_3$ is square-free, we may write

$$g_1 = p^{\alpha_1} g_1', \qquad g_2 = p^{\alpha_2} g_2', \qquad g_3 = p^{\alpha_3} g_3',$$

for $(\alpha_1, \alpha_2, \alpha_3) \in \{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}$ and $p \nmid g_i'$. Similarly, we write

$$h_2 = p^{\beta_2} h_2', \qquad h_3 = p^{\beta_3} h_3',$$

for $p \nmid h_2' h_3'$ and $\beta_3 \geqslant \beta_2 \geqslant 1$. Then one proceeds by considering solutions to the equation

$$p^{\alpha_1} g_1' h_1 w_1^2 + p^{\alpha_2 + \beta_2} g_2' h_2' w_2^2 + p^{\alpha_3 + \beta_3} g_3' h_3' w_3^2 = 0. \tag{2.2}$$

Suppose for the moment that we are examining solutions $(u, v, w) \in \mathbb{Z}^3$ to the equation

$$au^2 + p^\sigma bv^2 + p^\tau cw^2 = 0, \tag{2.3}$$

for $0 \leqslant \sigma \leqslant \tau$ and $p \nmid abc$. Then arguing along similar lines to the proof of [9, Theorem 2], we sketch how this implies that (u, v, w) lies on one of at most 2 sublattices of $\mathbb{Z}^3$, each of determinant $p^{\delta(\sigma,\tau)}$, where

$$\delta(\sigma, \tau) = \begin{cases} (\sigma + \tau) - 3\sigma/2, & \sigma \text{ even}, \\ (\sigma + \tau) - [3\sigma/2] + 1, & \sigma \text{ odd}. \end{cases} \tag{2.4}$$

Suppose first that $\sigma = 2s$ is even. Then (2.3) implies that $p^s \mid u$. By writing $u = p^s u'$, and considering the corresponding congruence $au'^2 + bv^2 \equiv 0 \pmod{p^{\tau-\sigma}}$, we therefore deduce that (u, v, w) lies on one of at most 2 integer lattices, each of determinant $p^{s+\tau-\sigma} = p^{\delta(\sigma,\tau)}$. Suppose now that $\sigma = 2s + 1$ is odd. In view of (2.3) we may again write $u = p^s u'$, and consider the corresponding congruence $au'^2 + pbv^2 \equiv 0 \pmod{p^{\tau-2s}}$. Since $\tau - 2s \geqslant 1$, we may clearly write $u' = pu''$, and so consider solutions to the equation

$$pa(u'')^2 + bv^2 + p^{\tau-\sigma} cw^2 = 0.$$

Now either $\tau - \sigma = 0$, or else we may write $v = pv'$ and consider the equation $a(u'')^2 + pbv'^2 + p^{\tau-\sigma-1} cw^2 = 0$. In the former case we conclude that (v, w) lies on one of at most two integer sublattices of determinant p. But then (u, v, w) lies on one of at most 2 integer lattices, each of determinant $p^{s+2} = p^{\delta(\sigma,\tau)}$. In the latter case we have $\tau - \sigma \geqslant 1$, and we proceed inductively. Thus either $\tau - \sigma = 1$, in which case we deduce that (u, v, w) lies on one of 2 integer lattices of determinant $p^{s+3} = p^{\delta(\sigma,\tau)}$, or else $\tau - \sigma \geqslant 2$ and we can repeat the process. Since this process clearly terminates we therefore deduce that whenever σ is odd, (u, v, w) lies on one of at most 2 integer lattices, each of determinant $p^{\delta(\sigma,\tau)}$.

Returning to (2.2), our goal is to show that $\mathbf{w}$ lies on one of at most 2 integer sublattices of $\mathbb{Z}^3$, each of determinant

$$\geqslant p^{\beta_2+\beta_3-[3(\alpha_1+\alpha_2+\alpha_3+\beta_2)/2]}. \tag{2.5}$$

In view of the existing proof of [9, Theorem 2], this will suffice to establish Lemma 2 since the inequalities $\beta_3 \geqslant \beta_2 \geqslant 1$ imply that

$$\nu_p(h_1h_2h_2) = \beta_2 + \beta_3, \qquad \nu_p(D_{\mathbf{g},\mathbf{h}}) = \alpha_1 + \alpha_2 + \alpha_3 + \beta_2.$$

Suppose first that $(\alpha_1, \alpha_2) = (0, 0)$. Then our work above shows that $\mathbf{w}$ lies on one of at most 2 integer sublattices of $\mathbb{Z}^3$, each of determinant

$$p^{\delta(\beta_2,\alpha_3+\beta_3)} \geqslant p^{\delta(\beta_2,\beta_3)}.$$

This is plainly satisfactory for (2.5), by (2.4). Suppose now that $(\alpha_1, \alpha_2, \alpha_3) = (1, 0, 0)$. If $\beta_2 = 0$, then it is not hard to conclude that $\mathbf{w}$ lies on one of at most 2 lattices of determinant p^{β_3}, which is also satisfactory. If now $\beta_2 \geqslant 1$ we obtain an equation of the shape (2.3), with $\sigma = \beta_2 - 1$ and $\tau = \beta_3 - 1$. Thus we obtain at most 2 integer lattices, each of determinant $p^{\delta(\beta_2-1,\beta_3-1)}$. It is easily checked that this quantity is bounded below by (2.5). Finally, we suppose that $(\alpha_1, \alpha_2, \alpha_3) =$

$(0,1,0)$. In this case we again obtain an equation of the form (2.3). Suppose first that $\beta_2 < \beta_3$, so that we may take $\sigma = \beta_2 + 1$ and $\tau = \beta_3$ in (2.3). But then it easily follows that $\mathbf{w}$ lies on one of at most 2 integer lattices, each of determinant

$$p^{\delta(\beta_2+1,\beta_3)} \geqslant p^{\beta_2+\beta_3-[3(1+\beta_2)/2]}.$$

Alternatively, if $\beta_2 = \beta_3 = \beta$ say, then we take $\sigma = \beta$ and $\tau = \beta + 1$ in (2.3), thereby deducing that $\mathbf{w}$ lies on one of at most 2 lattices, each of determinant

$$p^{\delta(\beta,\beta+1)} \geqslant p^{2\beta-[3(1+\beta)/2]}.$$

This completes the proof of Lemma 2.

2.2. *Solubility of quadratic forms*

In addition to considering the density of integer solutions to diagonal quadratic equations, as in the previous section, we shall also need to consider how often such an equation has at least one nontrivial integer solution. Let $\mathbf{a} \in Z_*^3$, let $Y_1, Y_2, Y_3 \geqslant 1$, and let $H \in \mathbb{N}$. We shall write $T(\mathbf{Y}; \mathbf{a}, H)$ to denote the set of pairwise coprime nonzero integers y_1, y_2, y_3 such that $|y_i| \leqslant Y_i$, and

$$\gcd(a_i y_i, a_j y_j) \mid H, \tag{2.6}$$

and such that the equation

$$a_1 y_1 x_1^2 + a_2 y_2 x_2^2 + a_3 y_3 x_3^2 = 0$$

has a nonzero solution $\mathbf{x} \in \mathbb{Z}^3$ with $\gcd(x_i, x_j) = 1$. If $\mathbf{y} \in T(\mathbf{Y}; \mathbf{a}, H)$ then it necessarily follows that

$$\left(\frac{-a_i a_j y_i y_j}{p}\right) \neq -1 \quad \text{for all odd primes } p \mid a_k y_k, \tag{2.7}$$

where $(\frac{n}{p})$ denotes the Legendre symbol for any $n \in \mathbb{Z}$ and odd prime p, and as usual $\{i, j, k\}$ denotes any permutation of the set $\{1, 2, 3\}$. Define the arithmetic function

$$\vartheta(n) = \prod_{p \mid n} \left(1 + \frac{1}{p}\right), \tag{2.8}$$

for any $n \in \mathbb{N}$. We shall proceed under the assumption that

$$Y_1 \leqslant Y_2 \leqslant Y_3. \tag{2.9}$$

With this in mind the goal of this section is to establish the following estimate.

Proposition 1. *Let $\varepsilon > 0$. Then we have*

$$\sum_{\mathbf{y}\in T(\mathbf{Y};\mathbf{a},H)} 2^{\omega(y_1y_2y_3)} \ll_\varepsilon \vartheta(a_1a_2)H^\varepsilon\big(Y_1Y_2Y_3 + (Y_1Y_2)^{1/2+\varepsilon}Y_3m_\varepsilon(\mathbf{a},\mathbf{Y})\big),$$

with

$$m_\varepsilon(\mathbf{a},\mathbf{Y}) = \min\big\{|a_1a_2|, Y_3\big\}^\varepsilon + \log Y_3. \tag{2.10}$$

Before proceeding with the proof of Proposition 1, it is interesting to place it in the context of other work in the literature. Let $T(Y)$ denote the set $T(\mathbf{Y};\mathbf{a},H)$ in the special case $Y_i = Y$ and $a_i = H = 1$, and let $T_*(Y)$ denote the set of $\mathbf{y}\in T(Y)$ for which the product $y_1y_2y_3$ is square-free. Then it follows from Proposition 1 that

$$\sum_{\mathbf{y}\in T_*(Y)} 2^{\omega(y_1y_2y_3)} \leqslant \sum_{\mathbf{y}\in T(Y)} 2^{\omega(y_1y_2y_3)} \ll Y^3. \tag{2.11}$$

This should be compared with the work of Guo [5] and Serre [14], who have both made a study of the corresponding sum weighted instead by 1. By using the large sieve inequality Serre has shown that

$$\#T(Y) \ll \frac{Y^3}{(\log Y)^{3/2}},$$

and Guo has proved an asymptotic formula for $\#T_*(Y)$ which agrees with this upper bound. In particular, together with (2.11), these estimates show that a random conic in $\mathbb{P}^2$ does not contain a rational point.

We shall need several technical results during the proof of Proposition 1, and it will be convenient to list them here. We begin by recording a basic estimate for the average orders of ϑ and ϑ^2, as given by (2.8), whose simple proof we include here for completeness.

Lemma 3. *Let $z \geqslant 1$. Then we have*

$$\sum_{n\leqslant z}\vartheta(n) \leqslant \sum_{n\leqslant z}\vartheta(n)^2 \ll z.$$

Proof. The first inequality is trivial, since $\vartheta(n)\geqslant 1$ for all $n\in\mathbb{N}$. To see the second inequality we note that

$$\sum_{n\leqslant z}\vartheta(n)^2 = \sum_{n\leqslant z}\bigg(\sum_{d\mid n}\frac{|\mu(d)|}{d}\bigg)^2 = \sum_{n\leqslant z}\sum_{[d_1,d_2]\mid n}\frac{|\mu(d_1)\mu(d_2)|}{d_1d_2}, \tag{2.12}$$

where $[d_1,d_2] = d_1d_2/\gcd(d_1,d_2)$ denotes the least common multiple of d_1 and d_2. But on writing $n = [d_1,d_2]e$ we easily deduce that

$$\sum_{n\leqslant z}\vartheta(n)^2 \leqslant \sum_{d_1,d_2=1}^\infty \sum_{e\leqslant z/[d_1,d_2]}\frac{1}{d_1d_2} \leqslant z\sum_{d_1,d_2=1}^\infty \frac{\gcd(d_1,d_2)}{d_1^2d_2^2} \ll z,$$

since

$$\sum_{d_1,d_2=1}^{\infty} \frac{\gcd(d_1,d_2)}{d_1^2 d_2^2} \leqslant \sum_{d_1,d_2=1}^{\infty} \sum_{k \mid d_1,d_2} \frac{k}{d_1^2 d_2^2} \leqslant \sum_{k,d_1',d_2'=1}^{\infty} \frac{1}{k^3 {d_1'}^2 {d_2'}^2} \ll 1.$$

This completes the proof of Lemma 3. □

We shall also need to apply the large sieve inequality in our work. The following version of the large sieve is due to Montgomery [11].

Lemma 4. *Let $z, N \geqslant 1$. Suppose that $S \subseteq \mathbb{Z} \cap [1, N]$, and that for every prime $p \leqslant z$ there exists $\sigma(p) \in [0, p)$ such that the image of S in $\mathbb{Z}/p\mathbb{Z}$ has $p - \sigma(p)$ elements. Then we have*

$$\#S \ll \frac{N + z^2}{G(z)}, \quad \text{with } G(z) = \sum_{n \leqslant z} \bigl|\mu(n)\bigr| \prod_{p \mid n} \frac{\sigma(p)}{p - \sigma(p)}.$$

During the course of this section we will need some standard character sum estimates. The following estimate is due to Heath-Brown [8, Corollary 4].

Lemma 5. *Let $M, N \in \mathbb{N}$, and let $a_1, \dots, a_M$ and $b_1, \dots, b_N$ be arbitrary complex numbers satisfying $|a_m|, |b_n| \leqslant 1$. Then*

$$\sum_{\substack{m \leqslant M \\ 2 \nmid m}} \sum_{n \leqslant N} a_m b_n \left(\frac{n}{m}\right) \ll_{\varepsilon} (MN)^{\varepsilon} \bigl(M^{1/2} N + M N^{1/2}\bigr),$$

for any $\varepsilon > 0$.

Next we recall the Pólya–Vinogradov inequality, which can be found in the work of Davenport [3, Chapter 23], for example.

Lemma 6. *Let $M, N \in \mathbb{N}$, and let χ be a nonprincipal character modulo q. Then*

$$\sum_{M \leqslant n \leqslant N} \chi(n) \ll q^{1/2} \log q.$$

We are now ready to commence our proof of Proposition 1 in earnest. It is important to note that in the course of this proof, as throughout our work, we shall follow common practice and allow the small positive constant ε to take different values at different parts of the argument. For any $\mathbf{a} \in Z_*^3$, and any $Y_1, Y_2, Y_3 \geqslant 1$, recall the definition of the set $T(\mathbf{Y}; \mathbf{a}, H)$ above. It will be convenient to write

$$\mathcal{T} = \mathcal{T}(\mathbf{Y}; \mathbf{a}, H) = \sum_{\mathbf{y} \in T(\mathbf{Y}; \mathbf{a}, H)} 2^{\omega(y_1 y_2 y_3)}, \tag{2.13}$$

with which notation our task is to establish the inequality

$$\mathcal{T} \ll_\varepsilon \vartheta(a_1a_2)H^\varepsilon\big(Y_1Y_2Y_3 + (Y_1Y_2)^{1/2+\varepsilon}Y_3 m_\varepsilon(\mathbf{a}, \mathbf{Y})\big),$$

where $m_\varepsilon(\mathbf{a}, \mathbf{Y})$ is given by (2.10). Our approach is based upon a fusion of the ideas used by Guo and Serre in their work on this problem. Recall assumption (2.9). Whenever Y_2 is much smaller in size than Y_3 we shall be in a position to apply Lemma 4 to estimate $\mathcal{T}$. In the alternative case, in which Y_3 is bounded above by a power of Y_2, we shall employ Guo's approach. We begin by studying the latter case.

2.2.1. Proof of Proposition 1: $Y_3 \leqslant Y_2^{10}$

As indicated by the title, our aim in this section is to establish Proposition 1 under the assumption that

$$Y_3 \leqslant Y_2^{10}. \tag{2.14}$$

Given any $n \in \mathbb{Z}$, it will be convenient to extend the definition of the Legendre symbol $(\frac{n}{p})$ to all primes p by setting $(\frac{n}{2}) = 0$. Our first step is the observation that

$$1 + \left(\frac{n}{p}\right) = \begin{cases} 2, & (\frac{n}{p}) = 1, \\ 0, & (\frac{n}{p}) = -1, \\ 1, & \text{otherwise}, \end{cases} \tag{2.15}$$

for any integer n and prime p. Recalling (2.7) and the fact that $\gcd(y_i, y_j) = 1$ for any $\mathbf{y} \in T(\mathbf{Y}; \mathbf{a}, H)$, we see that

$$\begin{aligned}
\mathcal{T} &= \sum_{\mathbf{y}\in T(\mathbf{Y};\mathbf{a},H)} 2^{\omega(y_1)+\omega(y_2)+\omega(y_3)} \\
&\ll_\varepsilon H^\varepsilon \sum_{\substack{|y_i|\leqslant Y_i \\ \gcd(y_i,y_j)=1}} \prod_{\{i,j,k\}=\{1,2,3\}} \prod_{p\mid y_k} \left(1 + \left(\frac{-a_ia_jy_iy_j}{p}\right)\right) \\
&\ll_\varepsilon H^\varepsilon \sum_{\substack{|y_i|\leqslant Y_i \\ \gcd(y_i,y_j)=1}} \sideset{}{^\sharp}\sum_{d_i\mid y_i} \big|\mu(d_1d_2d_3)\big| \prod_{\{i,j,k\}=\{1,2,3\}} \left(\frac{-a_ia_jy_iy_j}{d_k}\right),
\end{aligned}$$

where $\sum^\sharp$ denotes a summation over odd divisors $d_i \mid y_i$. Here we have used (2.6) to deduce that $2^{\omega(\gcd(y_k,a_ia_j))} \leqslant 2^{\omega(H)} \ll_\varepsilon H^\varepsilon$. We first show that the contribution arising from the case in which $d_2 = d_3 = 1$ is $O_\varepsilon(H^\varepsilon Y_1Y_2Y_3)$, which is satisfactory. But the contribution from this case is clearly

$$\ll_\varepsilon H^\varepsilon Y_1Y_2Y_3 + H^\varepsilon \sum_{\substack{|y_i|\leqslant Y_i \\ \gcd(y_i,y_j)=1}} \sideset{}{^\sharp}\sum_{\substack{d_1\mid y_1 \\ d_1\neq 1}} \big|\mu(d_1)\big| \left(\frac{-a_2a_3y_2y_3}{d_1}\right).$$

Using the Möbius function to pick out the condition $\gcd(y_3, y_1y_2) = 1$, we may therefore apply Lemma 6 to deduce that the right hand term is

$$\begin{aligned}
&\ll_\varepsilon H^\varepsilon \sum_{|y_2|\leqslant Y_2} \sum_{\substack{|d_1e_1|\leqslant Y_1\\ d_1\neq 1}} \big|\mu(d_1)\big| \left| \sum_{k|d_1e_1y_2} \sum_{|z_3|\leqslant Y_3/k} \mu(k)\left(\frac{kz_3}{d_1}\right)\right| \\
&\ll_\varepsilon H^\varepsilon \sum_{|y_2|\leqslant Y_2} \sum_{\substack{|d_1e_1|\leqslant Y_1\\ d_1\neq 1}} \big|\mu(d_1)\big| \sum_{k|d_1e_1y_2} \big|\mu(k)\big| \left| \sum_{|z_3|\leqslant Y_3/k} \left(\frac{z_3}{d_1}\right)\right| \\
&\ll_\varepsilon (HY_1Y_2)^\varepsilon \sum_{|y_2|\leqslant Y_2} \sum_{\substack{|d_1e_1|\leqslant Y_1\\ d_1\neq 1}} d_1^{1/2} \ll_\varepsilon (HY_1Y_2)^\varepsilon Y_1^{3/2} Y_2 \ll_\varepsilon H^\varepsilon Y_1Y_2Y_3,
\end{aligned}$$

if $\varepsilon > 0$ is sufficiently small. It therefore suffices to establish the estimate

$$\sum_{\substack{|y_i|\leqslant Y_i\\ \gcd(y_i,y_j)=1}} \sideset{}{^\sharp}\sum_{\substack{d_i|y_i\\ d_2d_3\neq 1}} \big|\mu(d_1d_2d_3)\big| \prod_{\{i,j,k\}=\{1,2,3\}} \left(\frac{-a_ia_jy_iy_j}{d_k}\right) \ll Y_1Y_2Y_3, \tag{2.16}$$

in order to complete the proof of Proposition 1 under the assumption (2.14).

Our main tool in the proof of (2.16) will be Lemma 5. Let $D_1, D_2, D_3 \geqslant 1$. We begin by estimating the contribution to the left-hand side of (2.16) from those values of d_i such that

$$D_i \leqslant d_i < 2D_i.$$

Let us write $\mathcal{T}(\mathbf{D})$ for this contribution. Ultimately we shall sum over dyadic intervals for $D_i \ll Y_i$ to deduce (2.16). Now for any permutation $\{i, j, k\}$ of $\{1, 2, 3\}$ we see that

$$\mathcal{T}(\mathbf{D}) \ll \sum_{|d_ie_i|\leqslant Y_i} \sum_{d_j\ll D_j} \sum_{e_k\ll Y_k/D_k} \left| \sideset{}{^\sharp}\sum_{D_k\leqslant d_k<2D_k} \sum_{e_j\ll Y_j/D_j} c(d_k, e_j)\left(\frac{e_j}{d_k}\right)\right|,$$

for certain coefficients $c(d_k, e_j) \in \mathbb{Z}$ having modulus at most 1. Hence it easily follows from Lemma 5 that for any $\varepsilon > 0$ we have

$$\begin{aligned}
\mathcal{T}(\mathbf{D}) &\ll_\varepsilon Y_3^\varepsilon \sum_{|d_ie_i|\leqslant Y_i} \sum_{d_j\ll D_j} \sum_{e_k\ll Y_k/D_k} \left(D_k^{1/2}\frac{Y_j}{D_j} + D_k\left(\frac{Y_j}{D_j}\right)^{1/2}\right) \\
&\ll_\varepsilon Y_3^\varepsilon \left(\frac{Y_iY_jY_k}{D_k^{1/2}} + Y_iY_j^{1/2}Y_kD_j^{1/2}\right).
\end{aligned} \tag{2.17}$$

This will be satisfactory when exactly one of D_2 or D_3 is large. To handle the case in which both D_2 and D_3 are large, we proceed by noting that

$$\mathcal{T}(\mathbf{D}) \ll \sum_{|d_1e_1|\leqslant Y_1} \sum_{\substack{e_2\ll Y_2/D_2\\ e_3\ll Y_3/D_3}} \left| \sideset{}{^\sharp}\sum_{d_2,d_3} c'(d_2, d_3)\left(\frac{d_2}{d_3}\right)\right|,$$

for certain coefficients $c'(d_2, d_3) \in \mathbb{Z}$ having modulus at most 1. But then an application of Lemma 5 immediately yields

$$\mathcal{T}(\mathbf{D}) \ll_\varepsilon \frac{Y_1 Y_2 Y_3^{1+\varepsilon}}{D_2 D_3}\big(D_2^{1/2} D_3 + D_2 D_3^{1/2}\big), \tag{2.18}$$

for any $\varepsilon > 0$. Finally we must consider the case in which both D_2 and D_3 are small. For this we recall that we have already handled the contribution from those d_2, d_3 for which $d_2 d_3 = 1$. Suppose first that $d_2 \neq 1$. Then Lemma 6 yields the contribution

$$\begin{aligned}
&\ll \sum_{d_3 \ll D_3} \sum_{\substack{e_1 \ll Y_1/D_1 \\ e_2 \ll Y_2/D_2}} \sum_{\substack{d_1 \ll D_1,\ d_2 \ll D_2 \\ d_2 \neq 1}} \big|\mu(d_1 d_2)\big| \left| \sum_{k \mid d_1 d_2 e_1 e_2} \mu(k) \sum_{f_3 \ll Y_3/(D_3 k)} \left(\frac{k f_3}{d_1 d_2}\right) \right| \\
&\ll_\varepsilon Y_2^\varepsilon \sum_{d_3 \ll D_3} \sum_{\substack{e_1 \ll Y_1/D_1 \\ e_2 \ll Y_2/D_2}} \sum_{d_1 \ll D_1,\ d_2 \ll D_2} (D_1 D_2)^{1/2} \\
&\ll_\varepsilon Y_1 Y_2^{1+\varepsilon} (D_1 D_2)^{1/2} D_3 \ll_\varepsilon Y_1 Y_2 Y_3^{1/2+\varepsilon} D_2^{1/2} D_3
\end{aligned}$$

to $\mathcal{T}(\mathbf{D})$, since $D_1 \ll Y_1 \leqslant Y_3$. Alternatively, if $d_2 = 1$ and $d_3 \neq 1$, then we obtain the contribution

$$\ll \sum_{\substack{e_1 \ll Y_1/D_1 \\ e_3 \ll Y_3/D_3}} \sum_{\substack{d_1 \ll D_1,\ d_3 \ll D_3 \\ d_3 \neq 1}} \big|\mu(d_1 d_3)\big| \left| \sum_{k \mid d_1 d_3 e_1 e_3} \mu(k) \sum_{f_2 \ll Y_2/k} \left(\frac{k f_2}{d_1 d_3}\right) \right| \ll_\varepsilon Y_1 Y_2^{1/2} Y_3^{1+\varepsilon} D_3^{1/2}$$

to $\mathcal{T}(\mathbf{D})$. Thus we may combine these two estimates to deduce that

$$\mathcal{T}(\mathbf{D}) \ll_\varepsilon Y_1 Y_2^{1/2} Y_3^{1/2+\varepsilon}\big(D_3^{1/2} Y_3^{1/2} + Y_2^{1/2} D_2^{1/2} D_3\big). \tag{2.19}$$

We are now in a position to collect together our various estimates to establish the bound

$$\mathcal{T}(\mathbf{D}) \ll_\varepsilon Y_1 Y_2^{7/8} Y_3^{1+\varepsilon}, \tag{2.20}$$

for any $\varepsilon > 0$. Before doing so we note that on summing over dyadic intervals for $D_i \ll Y_i$ this is enough to establish that the left-hand side of (2.16) is $O_\varepsilon(Y_1 Y_2^{7/8} Y_3^{1+\varepsilon})$. It is at this point that we employ assumption (2.14), which implies in particular that $\log Y_3 \ll \log Y_2$. This therefore establishes that

$$\mathcal{T} \ll_\varepsilon H^\varepsilon Y_1 Y_2 Y_3$$

provided that the value of ε is taken to be sufficiently small, and so completes the deduction of Proposition 1 from (2.20) under the assumption that (2.14) holds. In order to establish (2.20) we shall need to split the argument according to the size of D_2, D_3. On supposing first that $D_2, D_3 \ll Y_2^{1/4}$, it clearly follows from (2.19) that

$$\mathcal{T}(\mathbf{D}) \ll_\varepsilon Y_1 Y_2^{1/2} Y_3^{1/2+\varepsilon}\big(Y_2^{1/8} Y_3^{1/2} + Y_2^{7/8}\big) \ll_\varepsilon Y_1 Y_2^{7/8} Y_3^{1+\varepsilon},$$

which is satisfactory for (2.20). Similarly, if $D_2, D_3 \gg Y_2^{1/4}$, then (2.18) implies that (2.20) holds. Next we suppose that $D_2 \ll Y_2^{1/4}$ and $D_3 \gg Y_2^{1/4}$. But then we may apply (2.17) to the permutation $(i, j, k) = (1, 2, 3)$ to get

$$\mathcal{T}(\mathbf{D}) \ll_\varepsilon Y_3^\varepsilon \left(\frac{Y_1 Y_2 Y_3}{Y_2^{1/8}} + Y_1 Y_2^{5/8} Y_3 \right) \ll_\varepsilon Y_1 Y_2^{7/8} Y_3^{1+\varepsilon},$$

which is satisfactory for (2.20). Finally, if $D_2 \gg Y_2^{1/4}$ and $D_3 \ll Y_2^{1/4}$ then an application of (2.17) to the permutation $(i, j, k) = (1, 3, 2)$ also yields (2.20).

2.2.2. Proof of Proposition 1: $Y_3 > Y_2^{10}$

We now turn to the proof of Proposition 1 under the assumption that

$$Y_3 > Y_2^{10}. \tag{2.21}$$

In view of the previous section, this will suffice to complete the proof of Proposition 1. Note that $2^{\omega(n)} \leqslant d(n)$ for any $n \in \mathbb{N}$, where $d(n)$ is the ordinary divisor function. Then on recalling the conditions (2.6) and (2.7), we see that the quantity (2.13) satisfies

$$\begin{aligned} \mathcal{T} &\leqslant \sum_{\substack{|y_1| \leqslant Y_1,\ |y_2| \leqslant Y_2 \\ \gcd(y_1, y_2) = 1}} 2^{\omega(y_1 y_2)} \sum_{y_3:\ \mathbf{y} \in T(\mathbf{Y}; \mathbf{a}, H)} \sum_{d \mid y_3} 1 \\ &\ll \sum_{\substack{|y_1| \leqslant Y_1,\ |y_2| \leqslant Y_2 \\ \gcd(y_1, y_2) = 1}} 2^{\omega(y_1 y_2)} \sum_{d \leqslant Y_3^{1/2}}^{\dagger} \#\big\{e \in \mathbb{Z}:\ (y_1, y_2, de) \in T(\mathbf{Y}; \mathbf{a}, H)\big\}, \end{aligned}$$

where the summation $\sum^\dagger$ is only over integers $d \leqslant Y_3^{1/2}$ such that

$$\gcd(d, a_1 a_2) \mid H^2, \qquad \gcd(d, y_1 y_2) = 1, \tag{2.22}$$

and

$$\left(\frac{-a_1 a_2 y_1 y_2}{p} \right) = 1$$

for all odd primes p dividing $d / \gcd(d, a_1 a_2)$.

On defining the set

$$S = S(Y_3/d; y_1, y_2, \mathbf{a}, H) = \big\{e \in \mathbb{Z}:\ (y_1, y_2, de) \in T(\mathbf{Y}; \mathbf{a}, H)\big\},$$

so that, in particular,

$$\mathcal{T} \ll \sum_{\substack{|y_1| \leqslant Y_1, |y_2| \leqslant Y_2 \\ \gcd(y_1, y_2) = 1}} 2^{\omega(y_1 y_2)} \sum_{d \leqslant Y_3^{1/2}}^{\dagger} \#S, \tag{2.23}$$

we see that our task is now to estimate the size of S. For this we shall use the large sieve inequality, as presented in Lemma 4. For any odd prime p we must calculate the size of the image S_p of S in $\mathbb{Z}/p\mathbb{Z}$. If $p \nmid a_i y_1 y_2 d$ then the congruence

$$a_1 y_1 x_1^2 + a_2 y_2 x_2^2 + a_3 d e x_3^2 \equiv 0 \pmod p$$

is always soluble whenever $p \nmid e$, by the Chevalley–Warning theorem. Alternatively, if $p \mid e$ then this congruence is soluble if and only if

$$\left(\frac{-a_1 a_2 y_1 y_2}{p}\right) = 1. \tag{2.24}$$

Hence we conclude that

$$\#S_p = \begin{cases} p-1, & (\frac{-a_1 a_2 y_1 y_2}{p}) = -1, \\ p, & \text{otherwise,} \end{cases}$$

whenever $p \nmid a_i y_1 y_2 d$. Turning to the size of S_p in the case $p \mid a_i y_1 y_2$, we suppose that $p \mid a_1 y_1$ and $p \nmid a_2 a_3 y_2 d$. But then either $e = 0$, or else $e \neq 0$ and e belongs to S_p if and only if

$$\left(\frac{-a_2 a_3 y_2 d e}{p}\right) = 1.$$

Hence there are $\frac{1}{2}(p+1)$ possible values of e over all. Similarly one finds that $\#S_p = \frac{1}{2}(p+1)$ if $p \mid a_2 y_2$ and $p \nmid a_1 a_3 y_1 d$. If $p \mid a_3 d$ and $p \nmid a_1 a_2 y_1 y_2$ however, then $\#S_p = p$ since we may clearly suppose that (2.24) holds automatically for such primes. Finally, we note that $\#S_p = p$ in all other cases. Taking $\sigma(p)$ to be $p - \#S_p$, we have therefore shown that

$$\frac{\sigma(p)}{p-\sigma(p)} = \begin{cases} 1/(p-1), & p \nmid a_i y_1 y_2 d,\ (\frac{-a_1 a_2 y_1 y_2}{p}) = -1, \\ (p-1)/(p+1), & p \mid a_1 y_1,\ p \nmid a_2 a_3 y_2 d, \\ (p-1)/(p+1), & p \mid a_2 y_2,\ p \nmid a_1 a_3 y_1 d, \\ 0, & \text{otherwise.} \end{cases}$$

Now write $g(n) = |\mu(n)| \prod_{p \mid n} \frac{\sigma(p)}{p-\sigma(p)}$ for any $n \in \mathbb{N}$. Then

$$g(mn) = \begin{cases} g(m)g(n), & \text{if } \gcd(m,n) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

and $G(z) = \sum_{n \leqslant z} g(n)$. But then it is easy to deduce that

$$G(z) = \sum_{k \mid a_i y_1 y_2 d} g(k) \sum_{\substack{n \leqslant z/k \\ \gcd(n, a_i y_1 y_2 d) = 1}} g(n) \geqslant \sum_{k \mid y_1 y_2} g(k) \sum_{\substack{n \leqslant z/k \\ \gcd(n, a_i y_1 y_2 d) = 1}} g(n),$$

for any $z \geqslant 1$. We may now use the coprimality condition $\gcd(y_1, y_2) = 1$, together with (2.6) and (2.22) to conclude that

$$\sum_{k \mid y_1 y_2} g(k) = \prod_{p \mid y_1 y_2} \bigl(1 + g(p)\bigr) \gg_\varepsilon H^{-\varepsilon} 2^{\omega(y_1 y_2)} \prod_{p \mid y_1 y_2} \left(1 + \frac{1}{p}\right)^{-1},$$

for any $\varepsilon > 0$. Moreover it is not hard to deduce that

$$\begin{aligned} \sum_{\substack{n \leqslant z/k \\ \gcd(n, a_i y_1 y_2 d) = 1}} g(n) &\gg \sum_{\substack{n \leqslant z/k \\ p \mid n \Rightarrow (\frac{-a_1 a_2 y_1 y_2}{p}) = -1}} \frac{|\mu(n)|}{n} \\ &\gg \log(z/k) \Biggl(\sum_{\substack{n \leqslant z/k \\ p \mid n \Rightarrow (\frac{-a_1 a_2 y_1 y_2}{p}) \neq -1}} \frac{|\mu(n)|}{n} \Biggr)^{-1}. \end{aligned}$$

We may now complete our estimate for $\#S$ by taking $z = Y_3^{1/4}$ in Lemma 4. On noting that $Y_3/d \geqslant Y_3^{1/2}$, since $d \leqslant Y_3^{1/2}$, it therefore follows that

$$\#S \ll \frac{Y_3}{d G(Y_3^{1/4})}.$$

It is at this point that we apply the hypothesis (2.21), from which it follows that

$$Y_1 Y_2 \leqslant Y_2^2 < Y_3^{1/5}.$$

In particular, we see that $Y_3^{1/4}/k > Y_3^{1/20}$ for any divisor $k \mid y_1 y_2$. Recall the definition (2.8) of the arithmetic function ϑ. We therefore deduce that

$$G\bigl(Y_3^{1/4}\bigr) \gg_\varepsilon \frac{2^{\omega(y_1 y_2)}}{H^\varepsilon \vartheta(y_1 y_2)} \log Y_3 \Biggl(\sum_{\substack{n \leqslant Y_3^{1/2} \\ p \mid n \Rightarrow (\frac{-a_1 a_2 y_1 y_2}{p}) \neq -1}} \frac{|\mu(n)|}{n} \Biggr)^{-1},$$

for any $\varepsilon > 0$, whence

$$\#S \ll_\varepsilon \frac{H^\varepsilon \vartheta(a_1 a_2) \vartheta(y_1 y_2)^2}{2^{\omega(y_1 y_2)}} \frac{Y_3}{d \log Y_3} \sum_{\substack{n \leqslant Y_3^{1/2} \\ p \mid n \Rightarrow (\frac{-a_1 a_2 y_1 y_2}{p}) = 1}} \frac{1}{n}.$$

On inserting this into (2.23), we therefore deduce the statement of the following result.

Lemma 7. *Let $\varepsilon > 0$. Then we have*

$$\mathcal{T} \ll_\varepsilon \vartheta(a_1a_2)H^\varepsilon \frac{Y_3}{\log Y_3} \sum_{\substack{|y_1|\leqslant Y_1,\ |y_2|\leqslant Y_2\\ \gcd(y_1,y_2)=1}} \vartheta(y_1y_2)^2\bigg(\sum_{n\in\mathcal{N}} \frac{1}{n}\bigg)^2,$$

with

$$\mathcal{N} = \mathcal{N}(Y_3; \mathbf{a}, y_1, y_2) = \left\{n \in \mathbb{N}\colon\ n \leqslant Y_3^{1/2},\ \left(\frac{-a_1a_2y_1y_2}{p}\right) = 1 \text{ for all odd } p \mid n\right\}.$$

On writing $\mathcal{M} = \mathcal{N}(Y_3^2; \mathbf{a}, y_1, y_2)$, we observe that

$$\bigg(\sum_{n\in\mathcal{N}} \frac{1}{n}\bigg)^2 = \sum_{n_1,n_2\in\mathcal{N}} \frac{1}{n_1n_2} \leqslant \sum_{m\in\mathcal{M}} \frac{d(m)}{m},$$

in the statement of Lemma 7. But now we may clearly apply (2.15) in Lemma 7 to deduce that

$$\mathcal{T} \ll_\varepsilon \frac{\vartheta(a_1a_2)H^\varepsilon Y_3}{\log Y_3} \sum_{\substack{|y_1|\leqslant Y_1,\ |y_2|\leqslant Y_2\\ \gcd(y_1,y_2)=1}} \vartheta(y_1y_2)^2 \sum_{m\leqslant Y_3} \frac{1}{m} \sum_{d|m} \bigg(\frac{-a_1a_2y_1y_2}{d}\bigg). \tag{2.25}$$

We proceed by considering the contribution to (2.25) from the terms for which $d = k^2$ is a square. Writing $m = jk^2$, we obtain the contribution

$$\begin{aligned}
&\ll_\varepsilon \frac{\vartheta(a_1a_2)H^\varepsilon Y_3}{\log Y_3} \sum_{y_1,y_2} \vartheta(y_1)^2\vartheta(y_2)^2 \sum_{k\leqslant Y_3^{1/2}} \sum_{j\leqslant Y_3/k^2} \frac{1}{jk^2}\\
&\ll_\varepsilon \frac{\vartheta(a_1a_2)H^\varepsilon Y_3}{\log Y_3} \sum_{j\leqslant Y_3} \frac{1}{j} \sum_{y_1,y_2} \vartheta(y_1)^2\vartheta(y_2)^2 \ll_\varepsilon \vartheta(a_1a_2)H^\varepsilon Y_1Y_2Y_3,
\end{aligned}$$

by Lemma 3. This is plainly satisfactory for Proposition 1. In order to handle the contribution to (2.25) from the remaining divisors, we define the characteristic function

$$\delta(n) = \begin{cases} 0, & n = k^2 \text{ for some } k \in \mathbb{N},\\ 1, & \text{otherwise}, \end{cases}$$

for any $n \in \mathbb{N}$. Note that in particular we have $\delta(1) = 0$. Writing $m = de$, it follows that

$$\sum_{\substack{|y_1|\leqslant Y_1,\, |y_2|\leqslant Y_2\\ \gcd(y_1,y_2)=1}} \vartheta(y_1y_2)^2 \sum_{m\leqslant Y_3} \frac{1}{m} \sum_{\substack{d|m\\ \delta(d)=1}} \left(\frac{-a_1a_2y_1y_2}{d}\right) = \sum_{e\leqslant Y_3} \frac{S_e}{e}, \tag{2.26}$$

with

$$S_e = \sum_{d\leqslant Y_3/e} \frac{\delta(d)}{d} \sum_{\substack{|y_1|\leqslant Y_1,\, |y_2|\leqslant Y_2\\ \gcd(y_1,y_2)=1}} \vartheta(y_1)^2\vartheta(y_2)^2\bigg(\frac{-a_1a_2y_1y_2}{d}\bigg).$$

Our next task is to establish the following inequality.

Lemma 8. *Let $\varepsilon > 0$. Then we have*

$$S_e \ll_\varepsilon Y_1 Y_2 + (Y_1 Y_2)^{1/2+\varepsilon}\big(\min\{|a_1 a_2|, Y_3\}^\varepsilon + \log Y_3\big).$$

Proof. Let us consider the contribution $S_e(D_1, D_2)$ to S_e from d contained in the interval $D_1 \ll d \ll D_2$, for various choices of

$$1 \leqslant D_1 \leqslant D_2 \leqslant Y_3/e. \tag{2.27}$$

Suppose that $N \in \mathbb{Z}$ is not a square, and that $D \geqslant 1$. Then an application of Lemma 6 yields

$$\sum_{d \leqslant D} \delta(d)\left(\frac{N}{d}\right) = \sum_{d \leqslant D}\left(\frac{N}{d}\right) - \sum_{\substack{d \leqslant D \\ d = k^2}} 1 \ll N^{1/2}\log N + D^{1/2}.$$

Thus for any D_1, D_2 in the range (2.27), we may combine partial summation with Lemma 3 to deduce that the contribution to $S_e(D_1, D_2)$ from those y_1, y_2, for which $-a_1a_2y_1y_2$ is not a square, is

$$\begin{aligned}
&\ll \sum_{y_1, y_2} \delta(-a_1a_2y_1y_2)\vartheta(y_1)^2\vartheta(y_2)^2 \left| \sum_{D_1 \ll d \ll D_2} \frac{\delta(d)}{d}\left(\frac{-a_1a_2y_1y_2}{d}\right)\right| \\
&\ll_\varepsilon \sum_{y_1, y_2} \vartheta(y_1)^2\vartheta(y_2)^2\big(D_1^{-1/2} + D_1^{-1}|a_1a_2y_1y_2|^{1/2+\varepsilon}\big) \\
&\ll_\varepsilon Y_1Y_2 + D_1^{-1}|a_1a_2|^{1/2+\varepsilon}(Y_1Y_2)^{3/2+\varepsilon} \ll Y_1Y_2 + D_1^{-1}|a_1a_2|^{3/4}(Y_1Y_2)^{7/4},
\end{aligned}$$

on taking $\varepsilon = 1/4$. Now there are clearly $O((Y_1Y_2)^{1/2})$ values of y_1, y_2 for which $-a_1a_2y_1y_2$ is a square. In view of the trivial inequality $\vartheta(n) \ll_\varepsilon n^\varepsilon$ for any $\varepsilon > 0$, it follows that the total contribution to $S_e(D_1, D_2)$ from such y_1, y_2 is

$$\ll_\varepsilon (Y_1Y_2)^{1/2+\varepsilon}\log D_2$$

for any $\varepsilon > 0$. We have therefore established the bound

$$S_e(D_1, D_2) \ll_\varepsilon Y_1Y_2 + D_1^{-1}|a_1a_2|^{3/4}(Y_1Y_2)^{7/4} + (Y_1Y_2)^{1/2+\varepsilon}\log D_2, \tag{2.28}$$

for any $\varepsilon > 0$ and any D_1, D_2 in the range (2.27).

We shall need an alternative estimate for $S_e(D_1, D_2)$ to handle the contribution from small values of d. Let $[d_1, d_2]$ denote the least common multiple of d_1 and d_2, as usual. Then it follows from a combination of Lemma 6 and (2.12), that for fixed values of d, y_1 such that $\delta(d) = 1$, we have

$$\begin{aligned}\Bigg|\sum_{\substack{y_2\leqslant Y_2\\ \gcd(y_1,y_2)=1}}\vartheta(y_2)^2\left(\frac{y_2}{d}\right)\Bigg| &\leqslant \sum_{k|y_1}\big|\mu(k_1)\big|\vartheta(k)^2\Bigg|\sum_{y_2\leqslant Y_2/k}\vartheta(y_2)^2\left(\frac{y_2}{d}\right)\Bigg|\\ &\leqslant \sum_{k|y_1}\vartheta(k)^2\Bigg|\sum_{[d_1,d_2]|e|\leqslant Y_2/k}\frac{|\mu(d_1)\mu(d_2)|}{d_1d_2}\left(\frac{[d_1,d_2]e}{d}\right)\Bigg|\\ &\leqslant \sum_{k|y_1}\vartheta(k)^2\sum_{d_1,d_2\leqslant Y_2}\frac{1}{d_1d_2}\Bigg|\sum_{|e|\leqslant Y_2/(k[d_1,d_2])}\left(\frac{e}{d}\right)\Bigg|\\ &\ll_\varepsilon d^{1/2+\varepsilon}(Y_1Y_2)^\varepsilon.\end{aligned}$$

Hence a further application of Lemma 3 yields

$$\begin{aligned}S_e\big(1,(Y_1Y_2)^{1/2}\big) &\ll \sum_{|y_1|\leqslant Y_1}\vartheta(y_1)^2\sum_{d\leqslant(Y_1Y_2)^{1/2}}\frac{\delta(d)}{d}\Bigg|\sum_{\substack{y_2\leqslant Y_2\\ \gcd(y_1,y_2)=1}}\vartheta(y_2)^2\left(\frac{y_2}{d}\right)\Bigg|\\ &\ll_\varepsilon \sum_{|y_1|\leqslant Y_1}\vartheta(y_1)^2\sum_{d\leqslant(Y_1Y_2)^{1/2}}d^{-1/2+\varepsilon}(Y_1Y_2)^\varepsilon\\ &\ll_\varepsilon \sum_{|y_1|\leqslant Y_1}\vartheta(y_1)^2(Y_1Y_2)^{1/4+\varepsilon}\ll Y_1Y_2, \qquad (2.29)\end{aligned}$$

provided that $\varepsilon>0$ is taken to be sufficiently small.

For larger d, we employ Lemma 5. Suppose that $(Y_1Y_2)^{1/2}\leqslant D\leqslant Y_3/e$, and write $a(d)=\delta(d)(\frac{-a_1a_2}{d})$. Then proceeding as above we see that

$$S_e(D,D)=\sum_{D\ll d\ll D}\frac{a_d}{d}\sum_{\substack{[d_1,e_1]\|f_1|\leqslant Y_1\\ [d_2,e_2]\|f_2|\leqslant Y_2}}\frac{b(\mathbf{d},\mathbf{e},\mathbf{f})}{d_1d_2e_1e_2}\left(\frac{[d_1,e_1][d_2,e_2]f_1f_2}{d}\right),$$

where $b(\mathbf{d},\mathbf{e},\mathbf{f})=|\mu(d_1)\mu(d_2)\mu(e_1)\mu(e_2)|$ if $\gcd([d_1,e_1]f_1,[d_2,e_2]f_2)=1$ and $b(\mathbf{d},\mathbf{e},\mathbf{f})=0$ otherwise. In particular, we observe that $|a(d)b(\mathbf{d},\mathbf{e},\mathbf{f})|\leqslant 1$, and so there exists a certain constant $c(d,f_1,f_2)$ of modulus at most 1 such that

$$S_e(D,D)\ll\sum_{\substack{d_1,e_1\leqslant Y_1\\ d_2,e_2\leqslant Y_2}}\frac{1}{d_1d_2e_1e_2}\Bigg|\sum_{D\ll d\ll D}\frac{1}{d}\sum_{\substack{|f_1|\leqslant Y_1/[d_1,e_1]\\ |f_2|\leqslant Y_2/[d_2,e_2]}}c(d,f_1,f_2)\left(\frac{f_1f_2}{d}\right)\Bigg|.$$

On combining the fact that $Y_1Y_2\leqslant D^2$ with Lemma 5 we therefore deduce that

$$\begin{aligned}S_e(D,D) &\ll_\varepsilon \frac{(Y_1Y_2)^\varepsilon}{D^{1-\varepsilon}}\sum_{\substack{d_1,e_1\leqslant Y_1\\ d_2,e_2\leqslant Y_2}}\frac{1}{d_1d_2e_1e_2}\left(\frac{D^{1/2}Y_1Y_2}{[d_1,e_1][d_2,e_2]}+\frac{D(Y_1Y_2)^{1/2}}{([d_1,e_1][d_2,e_2])^{1/2}}\right)\\ &\ll_\varepsilon D^{\varepsilon-1/2}Y_1Y_2+D^\varepsilon(Y_1Y_2)^{1/2},\end{aligned}$$

for any $\varepsilon > 0$. Summing over dyadic intervals for $(Y_1Y_2)^{1/2} \leqslant D \leqslant D_2$, for any choice of $D_2 \leqslant Y_3/e$, we therefore obtain the estimate

$$S_e\big((Y_1Y_2)^{1/2}, D_2\big) \ll_\varepsilon Y_1Y_2 + D_2^\varepsilon (Y_1Y_2)^{1/2}.$$

Once combined with (2.29) this yields the bound

$$S_e(1, D_2) \ll_\varepsilon Y_1Y_2 + D_2^\varepsilon (Y_1Y_2)^{1/2}, \tag{2.30}$$

for any $(Y_1Y_2)^{1/2} \leqslant D_2 \leqslant Y_3/e$.

Taking $D_2 = Y_3/e$ in (2.30) we obtain the estimate

$$S_e \ll_\varepsilon Y_1Y_2 + (Y_1Y_2)^{1/2} Y_3^\varepsilon, \tag{2.31}$$

for any $\varepsilon > 0$. In order to obtain a second estimate, and so complete the proof of Lemma 8, we apply (2.30) with $D_2 = |a_1a_2|^{3/4}(Y_1Y_2)^{3/4}$ and (2.28) with $D_1 = |a_1a_2|^{3/4}(Y_1Y_2)^{3/4}$ and $D_2 = Y_3/e$. This produces the estimate

$$S_e \ll_\varepsilon Y_1Y_2 + (Y_1Y_2)^{1/2+\varepsilon}\big(|a_1a_2|^\varepsilon + \log Y_3\big),$$

for any $\varepsilon > 0$. We complete the proof of Lemma 8 by taking the minimum of the bounds provided by this inequality and (2.31). $\square$

It remains to substitute Lemma 8 into (2.26), and then insert the resulting estimate into (2.25). Thus it follows that

$$\begin{aligned} \mathcal{T} &\ll_\varepsilon \frac{\vartheta(a_1a_2)H^\varepsilon Y_3}{\log Y_3} \sum_{e \leqslant Y_3} \frac{S_e}{e} \\ &\ll_\varepsilon \vartheta(a_1a_2)H^\varepsilon Y_3\big(Y_1Y_2 + (Y_1Y_2)^{1/2+\varepsilon}\big(\min\big\{|a_1a_2|, Y_3\big\}^\varepsilon + \log Y_3\big)\big), \end{aligned}$$

which thereby establishes Proposition 1 under the assumption that (2.21) holds.

3. The equation $a_1b_1c_1^2 + a_2b_2c_2^2 + a_3b_3c_3^2 = 0$

The purpose of this section is to bring together the results in Sections 2.1 and 2.2, in order to make a study of the density of integer solutions to the equation

$$a_1b_1c_1^2 + a_2b_2c_2^2 + a_3b_3c_3^2 = 0. \tag{3.1}$$

Thus for any $A_i, B_i, C_i \geqslant 1$ we let $\mathcal{M}(A_i, B_i, C_i)$ denote the number of $\mathbf{a}, \mathbf{b}, \mathbf{c} \in Z_*^3$ such that (3.1) holds and

$$|a_i| \leqslant A_i, \qquad |b_i| \leqslant B_i, \qquad |c_i| \leqslant C_i,$$

with

$$\gcd(a_i, c_j) = \gcd(c_i, c_j) = 1 \tag{3.2}$$

and

$$\left|\mu(a_1a_2a_3)\right| = 1, \qquad \gcd(a_i, b_j, b_k) = 1. \tag{3.3}$$

It will be convenient to set

$$A = A_1A_2A_3, \qquad B = B_1B_2B_3, \qquad C = C_1C_2C_3.$$

With this notation in mind, we proceed by establishing the following result.

Proposition 2. *For any $\varepsilon > 0$, we have*

$$\mathcal{M}(A_i, B_i, C_i) \ll_\varepsilon A^{2/3}B^{2/3}C^{1/3} + \sigma\tau AB^{1/2}C^{1/2},$$

where

$$\sigma = 1 + \frac{\min\{A, B\}^\varepsilon}{\min\{B_iB_j\}^{1/16}}, \qquad \tau = 1 + \frac{\log B}{\min\{B_iB_j\}^{1/16}}.$$

Proof. We shall prove Proposition 2 by making suitable applications of Lemmas 1 and 2. Our starting point is to fix choices of $\mathbf{a}, \mathbf{c} \in Z_*^3$, and count the corresponding number of $\mathbf{b} \in Z_*^3$ satisfying (3.1) and $|b_i| \leqslant B_i$. Applying Lemma 1 with

$$\mathbf{h} = \left(a_1c_1^2, a_2c_2^2, a_3c_3^2\right),$$

we easily obtain the upper bound

$$\begin{aligned}\mathcal{M}(A_i, B_i, C_i) &\ll \sum_{\mathbf{a},\mathbf{c}}\left(1 + \frac{B}{\max|a_i|c_i^2B_i}\right) \leqslant \sum_{\mathbf{a},\mathbf{c}}\left(1 + B^{2/3}\left|a_1a_2a_3c_1^2c_2^2c_3^2\right|^{-1/3}\right) \\ &\ll AC + A^{2/3}B^{2/3}C^{1/3}.\end{aligned}$$

We shall use this bound whenever $C \leqslant B$, under which hypothesis the estimate in Proposition 2 clearly holds.

It remains to handle the case in which $C > B$. For this we fix choices of $\mathbf{a}, \mathbf{b} \in Z_*^3$ for which (3.3) holds, and count the corresponding number of $\mathbf{c} \in Z_*^3$ satisfying (3.1), (3.2) and $|c_i| \leqslant C_i$. Thus we are in a position to apply Lemma 2 with

$$\mathbf{g} = (a_1, a_2, a_3), \qquad \mathbf{h} = (b_1, b_2, b_3).$$

In particular, it follows from (3.1)–(3.3) that

$$\begin{aligned}D_{\mathbf{a},\mathbf{b}} &= \gcd(b_1b_2, b_1b_3, b_2b_3)\gcd(a_1, b_2b_3)\gcd(a_2, b_1b_3)\gcd(a_3, b_1b_2) = \gcd(b_1b_2, b_1b_3, b_2b_3) \\ &\leqslant \gcd(b_1, b_2)\gcd(b_1, b_3)\gcd(b_2, b_3) = E_{\mathbf{b}},\end{aligned}$$

say. Moreover, since $|b_1b_2b_3| \leqslant B$ and $C > B$, we also have

$$\frac{C}{|b_1b_2b_3|} \geqslant \frac{C}{B} > 1.$$

Thus under the assumption $C > B$ we may conclude from Lemma 2 that

$$\mathcal{M}(A_i, B_i, C_i) \ll C^{1/2} \sum_{\mathbf{a},\mathbf{b}}^{*} \frac{E_{\mathbf{b}}^{3/4}}{|b_1 b_2 b_3|^{1/2}} 2^{\omega(b_1 b_2 b_3)},$$

where $\sum_{\mathbf{a},\mathbf{b}}^{*}$ indicates a summation over $\mathbf{a}, \mathbf{b} \in Z_*^3$ for which $|a_i| \leqslant A_i$, $|b_i| \leqslant B_i$, (3.3) holds, and (3.1) has a solution $\mathbf{c} \in Z_*^3$ with (3.2) holding.

In order to handle the term $E_{\mathbf{b}}^{3/4}$ in our estimate for $\mathcal{M}(A_i, B_i, C_i)$, we write

$$b_i = h_{ij} h_{ik} b_i',$$

for fixed $h_{12}, h_{13}, h_{23} \in \mathbb{N}$ such that $h_{ij} = h_{ji}$. Then

$$|b_i'| \leqslant \frac{B_i}{h_{ij} h_{ik}} = B_i', \tag{3.4}$$

say. Since $\mathbf{b}$ is primitive, it follows that $\gcd(h_{ij}, h_{ik}) = 1$. Moreover, for fixed values of h_{12}, h_{13}, h_{23}, it suffices to sum over $\mathbf{a}, \mathbf{b}' \in Z_*^3$ for which

$$\gcd\bigl(h_{ij}, a_k b_k'\bigr) = \gcd\bigl(b_i', b_j'\bigr) = 1, \tag{3.5}$$

by (3.3) and the fact that $\mathbf{b}$ is primitive. With this change of variables, (3.1) clearly becomes

$$\beta_1 b_1' c_1^2 + \beta_2 b_2' c_2^2 + \beta_3 b_3' c_3^2 = 0, \tag{3.6}$$

where we have written

$$\beta_i = a_i h_{ij} h_{ik}$$

for fixed values of a_i, h_{ij}. We shall need to record the equality

$$\gcd\bigl(\beta_i b_i', \beta_j b_j'\bigr) = h_{ij}, \tag{3.7}$$

which easily follows from combining the coprimality conditions (3.2), (3.3), (3.5) and $\gcd(h_{ij}, h_{ik}) = 1$, with (3.6).

Write $h = h_{12} h_{13} h_{23}$ and suppose without loss of generality that

$$B_1' \leqslant B_2' \leqslant B_3'.$$

Then upon collecting our work together, we see that

$$\mathcal{M}(A_i, B_i, C_i) \ll C^{1/2} \sum_{h_{ij}} \sum_{\mathbf{a}} \sum_{\mathbf{b}' \in \mathcal{B}} \frac{2^{\omega(h^2 b_1' b_2' b_3')}}{h^{1/4} |b_1' b_2' b_3'|^{1/2}},$$

where $\mathcal{B} = \mathcal{B}(B_i; a_i, h_{ij})$ denotes the set of $\mathbf{b}' \in Z_*^3$ with pairwise coprime components, for which (3.4) and (3.7) hold, and (3.6) has a solution $\mathbf{c} \in Z_*^3$ with $\gcd(c_i, c_j) = 1$. Using the trivial upper bound $2^{\omega(n)} \ll_\varepsilon n^\varepsilon$, we may therefore combine Proposition 1 with Lemma 3 and partial summation to deduce that

$$\begin{aligned}\mathcal{M}(A_i, B_i, C_i) &\ll_\varepsilon C^{1/2} \sum_{h_{ij}} h^{\varepsilon-1/4} \sum_{\mathbf{a}} \sum_{\mathbf{b}' \in \mathcal{B}} \frac{2^{\omega(b_1' b_2' b_3')}}{|b_1' b_2' b_3'|^{1/2}} \\ &\ll_\varepsilon AC^{1/2} \sum_{h_{ij}} \frac{(B_1' B_2' B_3')^{1/2}}{h^{1/4-\varepsilon}} \left(1 + \frac{\min\{A, B\}^\varepsilon + \log B}{(B_1' B_2')^{1/2-\varepsilon}}\right).\end{aligned}$$

On recalling definition (3.4) of B_i', we see that

$$\sum_{h_{ij}} \frac{(B_1' B_2' B_3')^{1/2}}{h^{1/4-\varepsilon}} \ll B^{1/2},$$

provided that $\varepsilon > 0$ is taken to be sufficiently small. Similarly, using the inequalities

$$h_{12} \leqslant (B_1 B_2)^{1/2}, \qquad h_{13} \leqslant B_1 / h_{12}, \qquad h_{23} \leqslant B_2 / h_{12},$$

we easily check that

$$\sum_{h_{ij}} h^{\varepsilon-1/4} \left(B_1' B_2'\right)^\varepsilon {B_3'}^{1/2} \ll (B_1 B_2)^{3/8+\varepsilon} B_3^{1/2} \ll (B_1 B_2)^{7/16} B_3^{1/2},$$

provided that $\varepsilon > 0$ is sufficiently small. It therefore follows that the estimate in Proposition 2 holds in the case $C > B$, and so holds unconditionally. $\square$

It turns out that we shall need an alternative estimate for $\mathcal{M}(A_i, B_i, C_i)$ to handle the case in which B_1, B_2, B_3 have particularly awkward sizes. The following result is rather easy to establish.

Proposition 3. *We have*

$$\mathcal{M}(A_i, B_i, C_i) \ll A B_i B_j \left(C_k + C_i C_j A_k^{-1}\right) (\log AC)^2,$$

for any permutation $\{i, j, k\}$ *of the set* $\{1, 2, 3\}$.

Proof. Our proof of Proposition 3 is based upon Heath-Brown's treatment [10, Lemma 4] of the equation $n_1^2 n_2 n_3 + n_4^2 n_5 n_6 = n_7 n_8$. For fixed integers a, b, q we let $\rho(q; a, b)$ denote the number of solutions to the congruence $at^2 + b \equiv 0 \pmod{q}$. For any value of q, we then have

$$\rho(q; a, b) \leqslant \sum_{d \mid q} \left|\mu(d)\right| \left(\frac{-ab}{d}\right). \tag{3.8}$$

We shall establish Proposition 3 in the case $(i,j,k)=(1,2,3)$, say. The other cases will follow by symmetry. Now it follows from (3.1) that for given a_i, b_1, b_2, c_3, and each corresponding solution t of the congruence

$$a_1b_1t^2+a_2b_2\equiv 0 \pmod{a_3c_3^2},$$

we must have $c_1\equiv tc_2 \pmod{a_3c_3^2}$. This gives rise to an equation of the form $\mathbf{h}.\mathbf{w}=0$, with $\mathbf{h}=(1,-t,a_3c_3^2)$ and $\mathbf{w}=(c_1,c_2,k)$. Upon recalling that $\gcd(c_1,c_2)=1$ from (3.2), an application of Lemma 1 therefore yields the bound

$$\ll \rho\big(a_3c_3^2; a_1b_2, a_2b_2\big)\left(1+\frac{C_1C_2}{|a_3c_3^2|}\right),$$

for the number of possible b_3, c_1, c_2 given fixed choices for a_i, b_1, b_2 and c_3. On employing (3.8) we therefore have

$$\begin{aligned}\mathcal{M}(A_i,B_i,C_i) &\ll \sum_{a_i,b_1,b_2,c_3} \rho\big(a_3c_3^2; a_1b_2, a_2b_2\big)\left(1+\frac{C_1C_2}{|a_3c_3^2|}\right)\\ &\ll \sum_{a_i,b_1,b_2,c_3}\sum_{d\mid a_3c_3}\big|\mu(d)\big|\left(\frac{-a_1a_2b_1b_2}{d}\right)\left(1+\frac{C_1C_2}{|a_3c_3^2|}\right)\\ &\ll \sum_{a_i,b_1,b_2,c_3} d(a_3)d(c_3)\left(1+\frac{C_1C_2}{|a_3c_3^2|}\right),\end{aligned}$$

since the sum over square-free divisors of $a_3c_3^2$ is the same as the sum over square-free divisors of a_3c_3. But a simple application of partial summation now reveals that

$$\begin{aligned}\mathcal{M}(A_i,B_i,C_i) &\ll \sum_{a_i,b_1,b_2,c_3} d(a_3)d(c_3)+C_1C_2\sum_{a_i,b_1,b_2,c_3}\frac{d(a_3)d(c_3)}{|a_3c_3^2|}\\ &\ll (AB_1B_2C_3+A_1A_2B_1B_2C_1C_2)(\log AC)^2,\end{aligned}$$

which thereby completes the proof of Proposition 3. □

Although we shall not need to do so here, it is worth pointing out that with more work it is possible to remove the term $(\log AC)^2$ from the statement of Proposition 3.

4. Passage to the universal torsor

Our goal in this section is to equate the quantity (2.1) to the cardinality of a certain subset of integral points on the universal torsor above $\tilde{X}$. In fact our approach to the universal torsor rests upon an entirely elementary analysis of the equation defining the surface X, and we shall not prove here that the resulting parametrisation is actually the universal torsor above $\tilde{X}$. This fact will be supplied for us by the work of Hassett and Tschinkel [6].

In any solution $\mathbf{x}\in Z_*^4$ to the equation $F(\mathbf{x})=0$ we see that x_4 divides $x_1x_2x_3$. Hence we may write $x_4=y_1y_2y_3$ and $x_i=y_iz_i$, for some $\mathbf{y},\mathbf{z}\in\mathbb{Z}^3$ with $y_iz_i\neq 0$. Suppose that $z_i=\varepsilon_i z_i'$ for

$\varepsilon_i = \pm 1$ and $z_i' \in \mathbb{N}$. Then one easily employs the equation $F(\mathbf{x}) = 0$ to deduce that $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$. Hence, upon relabelling variables we may assume that

$$x_i = y_i z_i, \qquad x_4 = y_1 y_2 y_3,$$

for $(\mathbf{y}, \mathbf{z}) \in \mathbb{Z}^3 \times \mathbb{N}^3$ with $y_i \neq 0$.

Under this substitution, the equation $F(\mathbf{x}) = 0$ becomes

$$z_1 z_2 z_3 = (y_1 z_1 + y_2 z_2 + y_3 z_3)^2. \tag{4.1}$$

Since $\mathbf{x}$ is assumed to be primitive, it follows that $\mathbf{y}$ is primitive. Moreover, if p is any prime divisor of $\gcd(z_i, y_j)$, then (4.1) implies that p divides $y_k z_k$. But this contradicts the primitivity of $\mathbf{x}$, whence

$$\gcd(z_i, y_j) = \gcd(y_1, y_2, y_3) = 1. \tag{4.2}$$

We now write $z_i = w_i t_i^2$, for square-free $w_i \in \mathbb{N}$ and nonzero $t_i \in \mathbb{Z}$. In fact we may assume that $t_i \in \mathbb{N}$, since t_i and $-t_i$ produce the same value of z_i.

Next it follows from (4.2) that

$$\gcd(w_i, y_j) = \gcd(t_i, y_j) = 1, \tag{4.3}$$

and from (4.1) that $w_1 w_2 w_3$ is a square. Hence we can write

$$w_1 = u_2 u_3, \qquad w_2 = u_1 u_3, \qquad w_3 = u_1 u_2,$$

for square-free $u_i \in \mathbb{N}$, satisfying

$$\gcd(u_i, u_j) = \gcd(u_i, y_i) = 1. \tag{4.4}$$

Indeed w_i is square-free, and any prime divisor of $\gcd(u_i, y_i)$ must also divide $\gcd(w_j w_k, y_i)$, contrary to (4.3).

Substituting the quantities $w_i = u_j u_k$ into (4.1) therefore yields the expressions

$$\varepsilon t_1 t_2 t_3 u_1 u_2 u_3 = y_1 u_2 u_3 t_1^2 + y_2 u_1 u_3 t_2^2 + y_3 u_1 u_2 t_3^2, \tag{4.5}$$

where $\varepsilon = \pm 1$. It is clear that u_i must divide $y_i u_j u_k t_i^2$. But then u_i divides t_i, since u_i is square-free and $\gcd(u_i, y_i u_j u_k) = 1$, by (4.4). We proceed by writing

$$s_0 = \gcd(t_1/u_1, t_2/u_2, t_3/u_3),$$

and $s_i = t_i/(s_0 u_i)$. Plainly $s_0, s_i \in \mathbb{N}$, and $\mathbf{s} = (s_1, s_2, s_3)$ is primitive. Moreover, (4.3) yields

$$\gcd(u_i, y_j) = \gcd(s_i, y_j) = 1. \tag{4.6}$$

Substituting $t_i = s_0 s_i u_i$ into (4.5), we therefore obtain the equations

$$\varepsilon s_0 s_1 s_2 s_3 u_1 u_2 u_3 = y_1 u_1 s_1^2 + y_2 u_2 s_2^2 + y_3 u_3 s_3^2, \tag{4.7}$$

where $\varepsilon = \pm 1$, and $\gcd(s_0, y_i) = 1$ by (4.3).

We proceed by using this equation, together with the fact that $\mathbf{s}$ is primitive, to establish that

$$\gcd(s_i, u_j) = 1.$$

If p is any prime divisor of $\gcd(s_i, u_j)$ then it follows from (4.7), in conjunction with the coprimality conditions (4.4) and (4.6), that p divides s_k. Considering the corresponding p-adic order of each of the terms in (4.7), one is easily led to the conclusion that p divides $y_j s_j^2$, since u_j is square-free. But $\gcd(u_j, y_j) = 1$ by (4.4), and so p divides s_j, which is impossible. In fact we may go further and deduce that the components of $\mathbf{s}$ satisfy the relation

$$\gcd(s_i, s_j) = 1.$$

This follows immediately from (4.6), (4.7) and the fact that $\gcd(s_i, u_k) = 1$.

Let $\mathcal{T} \subset \mathbb{A}^{10}$ denote the set of $(s_0, \mathbf{s}, \mathbf{u}, \mathbf{y}) \in \mathbb{N} \times N^3 \times N^3 \times Z_*^3$ such that

$$s_0 s_1 s_2 s_3 u_1 u_2 u_3 = y_1 u_1 s_1^2 + y_2 u_2 s_2^2 + y_3 u_3 s_3^2 \tag{4.8}$$

with

$$\left|\mu(u_1 u_2 u_3)\right| = \gcd(s_i, s_j) = \gcd(s_i, u_j) = 1 \tag{4.9}$$

and

$$\gcd(s_0, y_i) = \gcd(s_i, y_j) = \gcd(u_i, y_1 y_2 y_3) = 1. \tag{4.10}$$

Now let $\mathbf{x} \in Z_*^4$ be any solution to the equation $F(\mathbf{x}) = 0$. Then tracing back through our argument, we deduce that there exists $(\pm s_0, \mathbf{s}, \mathbf{u}, \mathbf{y}) \in \mathcal{T}$ such that

$$x_i = y_i u_i^2 u_j u_k s_0^2 s_i^2, \qquad x_4 = y_1 y_2 y_3. \tag{4.11}$$

Conversely, given any $(\pm s_0, \mathbf{s}, \mathbf{u}, \mathbf{y}) \in \mathcal{T}$, the point given by (4.11) will be a primitive integer solution of the equation $F(\mathbf{x}) = 0$, with $x_1 x_2 x_3 x_4 \neq 0$. Indeed if p is any prime divisor of x_1, x_2, x_3, x_4 then we may assume that

$$p \mid y_i, \qquad p \mid s_0^2 u_1 u_2 u_3 \gcd\bigl(y_j u_j s_j^2, y_k u_k s_k^2\bigr).$$

But then (4.10) implies that $p \mid \gcd(y_1, y_2, y_3)$, which is impossible. We have therefore established the following result.

Lemma 9. *We have*

$$N_{U,H}(B) = \frac{1}{4}\#\bigl\{(s_0, \mathbf{s}, \mathbf{u}, \mathbf{y}) \in \mathcal{T}\colon \max\bigl\{\bigl|y_i u_i^2 u_j u_k s_0^2 s_i^2\bigr|, |y_1 y_2 y_3|\bigr\} \leqslant B\bigr\}.$$

Equation (4.8) is an affine embedding of the universal torsor above the minimal desingularisation $\tilde{X}$ of X. As already mentioned, it has been calculated by Hassett and Tschinkel [6, Section 4] by computing generators for the Cox ring $\mathrm{Cox}(\tilde{X})$ of $\tilde{X}$.

5. The lower bound

Our method of establishing the lower bound closely follows Heath-Brown's treatment of the Cayley cubic. Consequently we shall adopt similar notation throughout this section.

Let $P_1, P_2 \leqslant B^\delta$, for some suitably small choice of $\delta > 0$. This choice will be specified in (5.13), below. The idea is to fix choices of $\mathbf{s}, \mathbf{u} \in N^3$ such that (4.9) holds and

$$u_1u_2u_3 = P_1, \qquad s_1s_2s_3 = P_2.$$

In fact we shall insist upon the stronger condition that P_1P_2 is square-free. This is clearly permissible for the purposes of a lower bound. We then count the number of comparatively large nonzero solutions s_0, y_1, y_2, y_3 to the linear equation (4.8) subject to certain constraints.

Thus for $Y_0, Y_i \geqslant 1$, we let

$$\mathcal{N} = \mathcal{N}(\mathbf{s}, \mathbf{u}; Y_0, Y_1, Y_2, Y_3)$$

denote the number of $(s_0, y_1, y_2, y_3) \in \mathbb{N} \times \mathbb{Z}^3$ constrained by (4.8) and

$$\gcd(s_0, y_i) = \gcd(y_i, P_1P_2) = 1, \tag{5.1}$$

for which

$$Y_0 \leqslant s_0 < 2Y_0, \qquad Y_i \leqslant |y_i| < 2Y_i. \tag{5.2}$$

It should be clear that whenever (4.8) and (5.1) both hold, we automatically have (4.10) and $\gcd(y_1, y_2, y_3) = 1$. It will be convenient to define the quantities

$$A_0 = P_1P_2, \qquad A_i = u_is_i^2,$$

so that (4.8) may be written

$$A_0s_0 = A_1y_1 + A_2y_2 + A_3y_3. \tag{5.3}$$

Now it follows from Lemma 9 that we are only interested in values of s_0, y_i for which

$$A_0^2A_i\left|s_0^2y_i\right| \leqslant BP_1P_2^2, \qquad A_1A_2A_3|y_1y_2y_3| \leqslant BP_1P_2^2.$$

Hence we shall choose

$$Y_0 = \left[\frac{(BP_1P_2^2)^{1/3}}{2A_0}\right], \qquad Y_i = \left[\frac{(BP_1P_2^2)^{1/3}}{2A_i}\right]. \tag{5.4}$$

Much as in Heath-Brown's treatment, the main difficulty arises from having to keep track of the coprimality conditions (5.1). Let

$$Q = P_1P_2 \prod_{p \leqslant \sqrt{\log B}} p.$$

Following [10, (3.6)], we write

$$\mathcal{N} \geqslant \mathcal{N}_1 - \mathcal{N}_2, \tag{5.5}$$

where $\mathcal{N}_1$ is the number of solutions in which the condition $\gcd(s_0, y_i) = 1$ is replaced by the weaker condition

$$\gcd(s_0, y_i, Q) = 1,$$

and $\mathcal{N}_2$ is the number of solutions in which some y_i shares a prime factor p with s_0, such that $p \nmid Q$.

We proceed by estimating $\mathcal{N}_1$, for which we use the Möbius function to pick out the coprimality conditions. Let $\mathcal{N}_3(\mathbf{d}; \mathbf{e}) = \mathcal{N}_3(d_1, d_2, d_3; e_1, e_2, e_3)$ denote the number of solutions of (5.3) in the range (5.2) with $d_i \mid y_i$ and $e_i \mid s_0, y_i$. Then

$$\mathcal{N}_1 = \sum_{d_i \mid P_1 P_2} \mu(d_1)\mu(d_2)\mu(d_3) \sum_{e_i \mid Q} \mu(e_1)\mu(e_2)\mu(e_3)\mathcal{N}_3(\mathbf{d}; \mathbf{e}). \tag{5.6}$$

Our task is to estimate $\mathcal{N}_3(\mathbf{d}; \mathbf{e})$. Define the least common multiples

$$h_0 = [e_1, e_2, e_3], \qquad h_i = [d_i, e_i]$$

and the lattice

$$\Lambda = \bigl\{(n_1, n_2, n_3) \in \mathbb{Z}^3 \colon\ A_i h_i \mid n_i,\ A_0 h_0 \mid n_1 + n_2 + n_3\bigr\}.$$

Then upon defining the region

$$\mathcal{R} = \bigl\{\mathbf{r} \in \mathbb{R}^3 \colon\ A_i Y_i \leqslant |r_i| < 2A_i Y_i,\ A_0 Y_0 \leqslant |r_1 + r_2 + r_3| < 2A_0 Y_0\bigr\},$$

one follows the lines of Heath-Brown's argument in order to deduce that

$$\mathcal{N}_3(\mathbf{d}; \mathbf{e}) = \frac{\operatorname{vol}(\mathcal{R})}{\det \Lambda} + O\bigl((\det \Lambda)^2 \max\{Y_i, Y_0\}^2\bigr),$$

where

$$\det \Lambda = \frac{A_0 h_0 \prod_i A_i h_i}{\gcd(A_0 h_0, A_i h_i)}.$$

Since $d_i \mid P_1 P_2$ and $e_i \mid Q$, we deduce that $A_0 h_0 \leqslant P_1 P_2 Q^3$ and $A_i h_i \leqslant P_1^2 P_2^3 Q$. Hence we have

$$\det \Lambda \leqslant P_1^7 P_2^{10} Q^6 \ll P_1^{13} P_2^{16} \exp\bigl(O\bigl(\sqrt{\log B}\bigr)\bigr) \ll B^{30\delta}.$$

It follows that the error term in our estimate for $\mathcal{N}_3(\mathbf{d}; \mathbf{e})$ is $O(B^{2/3+62\delta})$, and so (5.6) becomes

$$\mathcal{N}_1 = \operatorname{vol}(\mathcal{R}) \sum_{d_i, e_i} \mu(d_1) \cdots \mu(e_3) \frac{\gcd(A_0 h_0, A_i h_i)}{A_0 h_0 \prod_i A_i h_i} + O\bigl(B^{2/3+63\delta}\bigr), \tag{5.7}$$

since there are at most $O(B^\delta)$ divisors of P_1P_2Q.

We now investigate the sum

$$\sum_{\substack{d_i \mid P_1P_2 \\ e_i \mid Q}} \mu(d_1)\cdots\mu(e_3)\frac{\gcd(A_0h_0, A_ih_i)}{A_0h_0\prod_i A_ih_i} = \prod_{p\mid Q} E_p, \tag{5.8}$$

say. When $p \mid Q$, but $p \nmid P_1P_2$, we see that

$$E_p = \sum_{\varepsilon_i \geqslant 0} \mu(p^{\varepsilon_1})\mu(p^{\varepsilon_2})\mu(p^{\varepsilon_3})\frac{\gcd(p^{\max\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}}, p^{\varepsilon_i})}{p^{\max\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}+\varepsilon_1+\varepsilon_2+\varepsilon_3}},$$

from which it easily follows that

$$E_p = 1 - \frac{3}{p^2} + \frac{2}{p^3}. \tag{5.9}$$

In this calculation we have used the fact that $p \nmid A_0A_i$ whenever $p \nmid P_1P_2$. Next, when $p \mid P_1$ we may assume that p divides precisely one factor, u_1 say. Since P_1P_2 is square-free it follows that $p \nmid u_2u_3P_2$, and that $p^2 \nmid u_1$. Let $A_0' = p^{-1}A_0$, $A_1' = p^{-1}A_1$, $A_2' = A_2$ and $A_3' = A_3$, so that

$$\frac{\gcd(A_0h_0, A_ih_i)}{A_0h_0\prod_i A_ih_i} = \frac{\gcd(pA_0'h_0, pA_1'h_1, A_2'h_2, A_3'h_3)}{p^2A_0'h_0\prod_i A_i'h_i},$$

with $p \nmid A_0'A_i'$. Then in this setting we see that

$$E_p = \frac{1}{p^2}\sum_{\delta_i,\varepsilon_i \geqslant 0} \mu(p^{\delta_1})\mu(p^{\delta_2})\cdots\mu(p^{\varepsilon_3})\frac{\gcd(p, p^{\max\{\delta_2,\varepsilon_2\}}, p^{\max\{\delta_3,\varepsilon_3\}})}{p^{\max\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}+\max\{\delta_1,\varepsilon_1\}+\cdots+\max\{\delta_3,\varepsilon_3\}}},$$

whence a straightforward calculation yields

$$E_p = \frac{1}{p^2}\left(1 - \frac{1}{p} - \frac{1}{p^2} + \frac{1}{p^3}\right). \tag{5.10}$$

Finally, we consider the case $p \mid P_2$, so that p divides precisely one factor, s_1 say. Since P_1P_2 is square-free it follows that $p \nmid P_1s_2s_3$, and that $p^2 \nmid s_1$. Let $A_0' = p^{-1}A_0$, $A_1' = p^{-2}A_1$, $A_2' = A_2$ and $A_3' = A_3$. Then arguing as above we now have

$$E_p = \frac{1}{p^3}\sum_{\delta_i,\varepsilon_i \geqslant 0} \mu(p^{\delta_1})\mu(p^{\delta_2})\cdots\mu(p^{\varepsilon_3})\frac{\gcd(p, p^{\max\{\delta_2,\varepsilon_2\}}, p^{\max\{\delta_3,\varepsilon_3\}})}{p^{\max\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}+\max\{\delta_1,\varepsilon_1\}+\cdots+\max\{\delta_3,\varepsilon_3\}}}.$$

In view of our calculation for (5.10) we immediately deduce that

$$E_p = \frac{1}{p^3}\left(1 - \frac{1}{p} - \frac{1}{p^2} + \frac{1}{p^3}\right). \tag{5.11}$$

Taking (5.9)– (5.11) together in (5.8), it therefore follows that

$$\sum_{d_i,e_i} \mu(d_1)\cdots\mu(e_3)\frac{\gcd(A_0h_0, A_ih_i)}{A_0h_0\prod_i A_ih_i} \gg \frac{1}{P_1^2P_2^3}\frac{\phi(P_1P_2)}{P_1P_2} = \frac{\phi(P_1P_2)}{P_1^3P_2^4},$$

since $\phi(n) = n\prod_{p|n}(1-1/p)$ for any $n \in \mathbb{N}$. Our choices (5.4) for Y_i, Y_0 clearly imply that $\mathrm{vol}(\mathcal{R}) \gg BP_1P_2^2$. We claim that

$$\mathcal{N}_1 \gg \frac{B}{P_1P_2}\frac{\phi(P_1P_2)}{P_1P_2}, \tag{5.12}$$

provided that we take

$$\delta = 1/201. \tag{5.13}$$

In order to establish the claim, it clearly suffices to check that the lower bound in (5.12) is larger than the error term in (5.7) when δ is taken to be $1/201$. But on using the trivial lower bound $\phi(n) \geqslant 1$ for any $n \in \mathbb{N}$, we see that

$$\frac{B}{P_1P_2}\frac{\phi(P_1P_2)}{P_1P_2} \geqslant \frac{B}{(P_1P_2)^2} \gg B^{1-4\delta}.$$

Since $B^{1-4\delta} \gg B^{2/3+63\delta}$ for $\delta = 1/201$, the claim follows.

Next we must produce an upper bound for $\mathcal{N}_2$, for which we may ignore any coprimality conditions whenever we wish to. Suppose that $p \mid s_0, y_1$, for some prime $p \nmid Q$ lying in the range $R \leqslant p < 2R$. In particular, we may assume that $R \ll Y_1$. There are $O(R)$ such primes, and we fix one particular choice. Following Heath-Brown's treatment, we write $s_0 = pt_0$ and $y_1 = pt_1$ and count solutions of the linear equation

$$pA_0t_0 = pA_1t_1 + A_2y_2 + A_3y_3. \tag{5.14}$$

In particular, t_0, t_1 are contained in the ranges

$$Y_0/R \ll |t_0| \ll Y_0/R, \qquad Y_1/R \ll |t_1| \ll Y_1/R.$$

Since P_1P_2 is square-free, it follows that $\gcd(A_i, A_0) = u_is_i$. Hence we may deduce from (5.14) that

$$A_2y_2 \equiv -A_3y_3 \pmod{pu_1s_1}.$$

We may assume by symmetry that $A_2Y_2 \geqslant A_3Y_3$. Upon noting that pu_1s_1 is coprime to A_2, since $\gcd(y_i, P_1P_2) = 1$, it follows that for each choice of y_3, there are $O(1 + Y_2/(Ru_1s_1))$ possibilities for y_2. Now (5.14) implies that

$$Ru_1s_1 \ll \max\{A_2Y_2, A_3Y_3\} = A_2Y_2. \tag{5.15}$$

Moreover, it follows from (5.4) that

$$A_2 u_1 s_1 \ll Y_2, \tag{5.16}$$

provided that $\delta \leqslant 1/5$. Together (5.15) and (5.16) imply that

$$1 \ll \left(\frac{A_2 Y_2}{R u_1 s_1}\right)^{1/2} \left(\frac{Y_2}{A_2 u_1 s_1}\right)^{1/2} = \frac{Y_2}{R^{1/2} u_1 s_1},$$

whence we deduce that there are $O(Y_2 Y_3/(R^{1/2} u_1 s_1))$ choices for y_2, y_3. We fix such a choice and write $A_2 y_2 + A_3 y_3 = p u_1 s_1 k$. Then it remains to count values of t_0, t_1 for which

$$u_2 u_3 s_2 s_3 t_0 = s_1 t_1 + k. \tag{5.17}$$

Now we have already seen that $R \ll Y_1$. Moreover, as in (5.16), we can use (5.4) to show that $u_2^3 u_3^3 s_2^3 s_3^3 \ll Y_1$ provided that $\delta \leqslant 1/15$. Together these inequalities imply that

$$1 \ll \left(\frac{Y_1}{R}\right)^{2/3} \left(\frac{Y_1}{u_2^3 u_3^3 s_2^3 s_3^3}\right)^{1/3} = \frac{Y_1}{R^{2/3} u_2 u_3 s_2 s_3}.$$

Viewing (5.17) as a congruence modulo $u_2 u_3 s_2 s_3$, one easily concludes that there are $O(Y_1/(R^{2/3} u_2 u_3 s_2 s_3))$ possibilities for t_0, t_1.

In conclusion we have therefore shown that the total number of admissible p, y_2, y_3, t_0, t_1, for which $R \leqslant p < 2R$, is

$$\ll R \cdot \frac{Y_2 Y_3}{R^{1/2} u_1 s_1} \cdot \frac{Y_1}{R^{2/3} u_2 u_3 s_2 s_3} \ll \frac{B P_1 P_2^2}{R^{1/6} P_1^2 P_2^3} = \frac{B}{R^{1/6} P_1 P_2},$$

by (5.4). Summing $R \gg \sqrt{\log B}$ over dyadic intervals, we deduce that

$$\mathcal{N}_2 \ll \frac{B}{P_1 P_2} (\log B)^{-1/12},$$

provided that $\delta \leqslant 1/15$. It follows from (5.12) and (5.13) that $\mathcal{N}_2 = o(\mathcal{N}_1)$, and so (5.5) implies that

$$\mathcal{N} \gg \frac{B}{P_1 P_2} \frac{\phi(P_1 P_2)}{P_1 P_2}.$$

Finally, in order to complete the proof of the lower bound in (1.1), we note that any square-free value of P will factorise into values $u_1, u_2, u_3, s_1, s_2, s_3$ satisfying (4.9), in precisely $d_6(P)$ ways. It therefore follows that

$$N_{U,H}(B) \gg \sum_{P \leqslant B^{2/201}} |\mu(P)| d_6(P) \frac{B}{P} \frac{\phi(P)}{P}. \tag{5.18}$$

To handle this quantity we define the sum

$$S(x)=\sum_{n\leqslant x}\frac{|\mu(n)|d_6(n)\phi(n)}{n},$$

for any $x>1$, and proceed by establishing the following simple bound.

Lemma 10. *For any $x>1$ we have*

$$S(x)\gg x(\log x)^5.$$

Proof. To establish the lemma we shall apply Perron's formula to the corresponding Dirichlet series

$$F(s)=\sum_{n=1}^{\infty}\frac{|\mu(n)|d_6(n)\phi(n)/n}{n^s},$$

defined for $\Re e(s)>1$. It is a trivial matter to calculate the Euler product

$$F(s)=\prod_p\left(1+\frac{6(1-1/p)}{p^s}\right)=\zeta(s)^6G(s),$$

for some function $G(s)$ that is holomorphic and bounded on the half-plane $\Re e(s)>1/2$. Let $\varepsilon>0$ and let $T\in[1,x]$. Then Perron's formula yields

$$S(x)=\frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT}\zeta(s)^6G(s)\frac{x^s}{s}\,\mathrm{d}s+O_\varepsilon\left(\frac{x^{1+\varepsilon}}{T}\right).$$

We apply Cauchy's residue theorem to the rectangular contour joining the points $2/3-iT$, $2/3+iT$, $1+\varepsilon+iT$ and $1+\varepsilon-iT$, which therefore leads to the conclusion that there exists a polynomial f of degree 5 such that

$$S(x)-xf(\log x)\ll_\varepsilon\frac{x^{1+\varepsilon}}{T}+\left(\int_{2/3-iT}^{2/3+iT}+\int_{2/3-iT}^{1+\varepsilon-iT}+\int_{1+\varepsilon+iT}^{2/3+iT}\right)\left|\zeta(s)^6\frac{x^s}{s}\right|\mathrm{d}s.$$

Here we have used the fact that $G(s)$ is bounded for $\Re e(s)>1/2$. To estimate this error term we apply the well-known convexity bound $\zeta(\sigma+it)\ll_\varepsilon|t|^{(1-\sigma)/3+\varepsilon}$, valid for any $\sigma\in[1/2,1]$ and $|t|\geqslant 1$. Thus it follows that

$$S(x)-xf(\log x)\ll_\varepsilon\frac{x^{1+\varepsilon}}{T}+x^{2/3+\varepsilon}T^{2/3}.$$

Selecting $T=x^{1/5}$ therefore completes the proof of Lemma 10. $\square$

On combining Lemma 10 with an application of partial summation, and then inserting the resulting estimate into (5.18), we therefore deduce that

$$N_{U,H}(B) \gg B(\log B)^6.$$

This completes the proof of the lower bound in (1.1).

6. The upper bound

Fix a choice of $X_1, \dots, X_4, S_0, S_i, U_i, Y_i \geqslant 1$. We shall write

$$\mathcal{N} = \mathcal{N}(X_1, \dots, X_4; S_0; S_1, S_2, S_3; U_1, U_2, U_3; Y_1, Y_2, Y_3)$$

for the total contribution to $N_{U,H}(B)$ from $\mathbf{x}$ contained in the intervals

$$X_\xi \leqslant |x_\xi| < 2X_\xi, \quad 1 \leqslant \xi \leqslant 4, \tag{6.1}$$

and such that the variables $s_0, \mathbf{s}, \mathbf{u}, \mathbf{y}$ appearing in Lemma 9 satisfy

$$S_0 \leqslant s_0 < 2S_0, \qquad S_i \leqslant |s_i| < 2S_i, \qquad U_i \leqslant |u_i| < 2U_i, \qquad Y_i \leqslant |y_i| < 2Y_i. \tag{6.2}$$

It will be convenient to relabel the indices so that

$$X_1 \leqslant X_2 \leqslant X_3. \tag{6.3}$$

Suppose that $\mathbf{x} \in Z_*^4$ is a solution of $F(\mathbf{x}) = 0$, with $|x_1|, \dots, |x_4| \leqslant B$. Then (6.1) implies that

$$X_1, X_2, X_3, X_4 \leqslant B. \tag{6.4}$$

If $\mathcal{N} = 0$ there is nothing to prove, and so we assume henceforth that the dyadic ranges in (6.1) and (6.2) produce a nonzero value of $\mathcal{N}$.

We proceed by showing that under the assumption that $\mathcal{N} \neq 0$, certain choices of dyadic ranges in (6.1) and (6.2) force certain other ranges to have fixed order of magnitude. It will be convenient to write

$$S = S_1S_2S_3, \qquad U = U_1U_2U_3, \qquad Y = Y_1Y_2Y_3.$$

Hence it follows from (4.11) that

$$X_i \ll Y_iU_iUS_0^2S_i^2 \ll X_i, \tag{6.5}$$

and that

$$X_4 \ll Y \ll X_4. \tag{6.6}$$

Together, (6.5) and (6.6) imply that

$$\left(\frac{X_1X_2X_3}{X_4}\right)^{1/2} \ll S_0^3SU^2 \ll \left(\frac{X_1X_2X_3}{X_4}\right)^{1/2}. \tag{6.7}$$

We take a moment to record two further inequalities satisfied by the quantities S_0, S_i, U_i, Y_i, which will be crucial in our final analysis. First we deduce from (6.5)–(6.7) that

$$S_0 S^{1/3} U^{2/3} Y^{2/3} = \left(S_0^3 S U^2\right)^{1/3} Y^{2/3} \ll (X_1 X_2 X_3)^{1/6} X_4^{1/2}. \tag{6.8}$$

Similarly, we may deduce that

$$S_0 S^{1/2} U Y^{1/2} \leqslant \left(S_0^3 S U^2\right)^{1/2} Y^{1/2} \ll (X_1 X_2 X_3 X_4)^{1/4}. \tag{6.9}$$

It is clear that $\mathcal{N}$ is bounded above by the number of $s_0 \in \mathbb{N}$ and $\mathbf{s}, \mathbf{u}, \mathbf{y} \in Z_*^3$ contained in the ranges (6.2), for which (4.8)–(4.10) all hold. Ultimately we shall sum over suitable dyadic intervals for X_1, X_2, X_3, X_4 and S_0, S_i, U_i, Y_i, in order to establish the upper bound in (1.1), and so complete the proof of the theorem.

For any fixed choice of $s_0 \in \mathbb{N}$ and $\mathbf{s}, \mathbf{u} \in Z_*^3$ in the region (6.2), with (4.9) holding, we let

$$N(s_0, \mathbf{s}, \mathbf{u}) = N(Y_1, Y_2, Y_3; s_0, \mathbf{s}, \mathbf{u})$$

denote the corresponding contribution to $\mathcal{N}$ from the $\mathbf{y} \in Z_*^3$. Clearly we are only interested in values of $s_0, \mathbf{s}, \mathbf{u}$ for which $N(s_0, \mathbf{s}, \mathbf{u})$ is nonzero. Considering $s_0, \mathbf{s}, \mathbf{u}$ to be fixed, we select any vector

$$\hat{\mathbf{y}} = (\hat{y}_1, \hat{y}_2, \hat{y}_3) \tag{6.10}$$

for which the Euclidean norm $|\hat{\mathbf{y}}|$ is least. Following the convention that this vector too is fixed, for fixed values of $s_0, \mathbf{s}, \mathbf{u}$, we define the change of variables

$$z_i = y_i - \hat{y}_i. \tag{6.11}$$

We shall let $N_1(s_0, \mathbf{s}, \mathbf{u}; \hat{\mathbf{y}})$ denote the overall contribution to $N(s_0, \mathbf{s}, \mathbf{u})$ from those $\mathbf{y}$ for which $z_1 z_2 z_3 \neq 0$, and we let $N_2(s_0, \mathbf{s}, \mathbf{u}; \hat{\mathbf{y}})$ denote the remaining contribution to $N(s_0, \mathbf{s}, \mathbf{u})$ from those $\mathbf{y}$ for which $z_1 z_2 z_3 = 0$. With this notation we therefore have

$$\mathcal{N} = \sum_{s_0, \mathbf{s}, \mathbf{u}} N(s_0, \mathbf{s}, \mathbf{u}) = \sum_{s_0, \mathbf{s}, \mathbf{u}} N_1(s_0, \mathbf{s}, \mathbf{u}; \hat{\mathbf{y}}) + \sum_{s_0, \mathbf{s}, \mathbf{u}} N_2(s_0, \mathbf{s}, \mathbf{u}; \hat{\mathbf{y}}) = \mathcal{N}_1 + \mathcal{N}_2, \tag{6.12}$$

say. Here the summations are over all $(s_0, \mathbf{s}, \mathbf{u}) \in \mathbb{N} \times Z_*^3 \times Z_*^3$ in the region (6.2), with (4.9) holding. It will be necessary to investigate the quantities $\mathcal{N}_1$ and $\mathcal{N}_2$ separately. Finally, we shall conclude that

$$N_{U,H}(B) \ll \sum_{S_0, S_i, U_i, Y_i, X_i, X_4} \mathcal{N}, \tag{6.13}$$

where the summation is over all dyadic intervals for X_1, X_2, X_3, X_4, subject to (6.4), and also all dyadic intervals for S_0, S_i, U_i, Y_i, subject to (6.5) and (6.6). Our first task is to show that the overall contribution from $\mathcal{N}_1$ to $N_{U,H}(B)$ is satisfactory.

Proposition 4. *We have*

$$\sum_{S_0,S_i,U_i,Y_i,X_i,X_4} \mathcal{N}_1 \ll B(\log B)^6,$$

where the summation is over dyadic intervals subject to (6.4)–(6.6).

Proposition 4 will be established in Section 6.1. Next in Section 6.2 we shall estimate the corresponding contribution from $\mathcal{N}_2$ to $N_{U,H}(B)$ via the following result.

Proposition 5. *We have*

$$\sum_{S_0,S_i,U_i,Y_i,X_i,X_4} \mathcal{N}_2 \ll B(\log B)^6,$$

where the summation is over dyadic intervals subject to (6.4)–(6.6).

Once taken together in (6.12) and (6.13), Propositions 4 and 5 therefore yield the upper bound

$$N_{U,H}(B) \ll B(\log B)^6,$$

which thereby completes the proof of (1.1).

6.1. Proof of Proposition 4

Our first step in the proof of Proposition 4 is to provide a pair of upper bounds for $\mathcal{N}_1$. For any fixed choice of $s_0 \in \mathbb{N}$ and $\mathbf{s}, \mathbf{u} \in Z_*^3$ in the region (6.2), with (4.9) holding, we let $\hat{\mathbf{y}}$ be the corresponding vector (6.10) that was selected above. On recalling the change of variables (6.11), it therefore follows from (4.8) that

$$z_1u_1s_1^2 + z_2u_2s_2^2 + z_3u_3s_3^2 = 0 \tag{6.14}$$

and from (6.2) that $|z_i| < 4Y_i$. Hence we deduce that

$$\mathcal{N}_1 \ll S_0 \sum_{\mathbf{s},\mathbf{u}} \#\{\mathbf{z} \in \mathbb{Z}^3\colon\ z_1z_2z_3 \neq 0,\ |z_i| < 4Y_i,\ (6.14)\text{ holds}\}.$$

Recall that $u_1u_2u_3$ is square-free, so that $\gcd(u_i, u_j) = 1$. It is apparent that the $\mathbf{z}$ appearing in the summand need not be primitive. Moreover we no longer necessarily have coprimality conditions corresponding to (4.10). In order to recover a weaker set of coprimality relations, we shall write

$$u_i = d_{jk}u_i', \qquad z_i = d_{ij}d_{ik}ez_i',$$

say, for any $d_{ij}, e \in \mathbb{N}$ with the convention that $d_{ij} = d_{ji}$. Let

$$U_i' = \frac{U_i}{d_{jk}}, \qquad Y_i' = \frac{Y_i}{d_{ij}d_{ik}e}$$

and

$$U' = U_1'U_2'U_3', \qquad Y' = Y_1'Y_2'Y_3'.$$

In particular, it follows from (4.9) that

$$\left|\mu\left(u_1'u_2'u_3'\right)\right| = \gcd(s_i, s_j) = \gcd\left(s_i, u_j'\right) = 1, \tag{6.15}$$

for any $\mathbf{s}$ and $\mathbf{u}'$.

For fixed values of $d_{ij}, e \in \mathbb{N}$, our task is to estimate the number of $\mathbf{s}, \mathbf{u}', \mathbf{z}' \in Z_*^3$ such that

$$\gcd\left(u_i', z_j', z_k'\right) = 1,$$

(6.15) holds,

$$S_i \leqslant |s_i| < 2S_i, \qquad U_i' \leqslant \left|u_i'\right| < 2U_i', \qquad \left|z_i'\right| < 4Y_i'$$

and

$$z_1'u_1's_1^2 + z_2'u_2's_2^2 + z_3'u_3's_3^2 = 0.$$

But this quantity is clearly bounded above by $\mathcal{M} = \mathcal{M}(2U_i', 4Y_i', 2S_i)$ in the notation of Section 3. Thus it follows that

$$\mathcal{N}_1 \ll S_0 \sum_{d_{ij}, e \ll Y} \mathcal{M}, \tag{6.16}$$

and Proposition 2 yields

$$\mathcal{M} \ll_\varepsilon S^{1/3}U'^{2/3}Y'^{2/3} + \sigma\tau S^{1/2}U'Y'^{1/2}, \tag{6.17}$$

for any $\varepsilon > 0$, where

$$\sigma = 1 + \frac{\min\{U, Y\}^\varepsilon}{\min\{Y_i'Y_j'\}^{1/16}}, \qquad \tau = 1 + \frac{\log Y}{\min\{Y_i'Y_j'\}^{1/16}}.$$

On applying Proposition 3 we obtain the alternative estimate

$$\mathcal{M} \ll U'Y_i'Y_j'\left(S_k + S_iS_jU_k'^{-1}\right)(\log SU')^2, \tag{6.18}$$

for any permutation $\{i, j, k\}$ of the set $\{1, 2, 3\}$.

We may now use (6.17) and (6.18) to obtain a pair of estimates for $\mathcal{N}_1$. Recall inequality (6.16) for $\mathcal{N}_1$, and note that

$$U' = (d_{12}d_{13}d_{23})^{-1}U, \qquad Y' = (d_{12}d_{13}d_{23})^{-2}e^{-3}Y.$$

Beginning with an application of (6.17), we deduce that

$$\sum_{d_{ij},e\ll Y} \mathcal{M} \ll_\varepsilon S^{1/3}U^{2/3}Y^{2/3} + \sigma\tau S^{1/2}UY^{1/2},$$

where

$$\sigma = 1 + \frac{\min\{U,Y\}^\varepsilon}{\min\{Y_iY_j\}^{1/16}}, \qquad \tau = 1 + \frac{\log B}{\min\{Y_iY_j\}^{1/16}}. \tag{6.19}$$

We therefore obtain the following estimate.

Lemma 11. *We have*

$$\mathcal{N}_1 \ll_\varepsilon S_0S^{1/3}U^{2/3}Y^{2/3} + \sigma\tau S_0S^{1/2}UY^{1/2},$$

for any $\varepsilon > 0$, where σ, τ are given by (6.19).

Similarly, since (6.4) and (6.5) imply that $(\log SU) \ll \log B$, an application of (6.18) yields the following alternative estimate.

Lemma 12. *We have*

$$\mathcal{N}_1 \ll S_0UY_iY_j\bigl(S_k + S_iS_jS_k^{-1}U_k^{-1}\bigr)(\log B)^2,$$

for any permutation $\{i,j,k\}$ of the set $\{1,2,3\}$.

We are now ready to complete the proof of Proposition 4. Now it follows from inequalities (6.2), (6.3) and (6.5) that

$$Y_1U_1S_1^2 \ll Y_2U_2S_2^2 \ll Y_3U_3S_3^2. \tag{6.20}$$

In particular, (4.8) implies that

$$S_0SU \ll Y_3U_3S_3^2. \tag{6.21}$$

Multiplying both sides of this inequality by S_0^2U, and recalling (6.4) and (6.5), we deduce that

$$S_0^3SU^2 \ll B. \tag{6.22}$$

It will also be useful to deduce an inequality involving the maximum size of the U_i. Suppose temporarily that $U_1 \leqslant U_2 \leqslant U_3$, so that $U_1U_2 \leqslant U_3^2$. Then in view of (6.4) and (6.5) it clearly follows that $U_1U_2U_3^2 \ll B$, whence $U_1U_2 \ll B^{1/2}$. Using this sort of argument it is not hard to deduce that, in general,

$$U_iU_j \ll B^{1/2}. \tag{6.23}$$

Throughout the proof of Proposition 4 we shall make the additional assumption that

$$Y_\imath \leqslant Y_\jmath \leqslant Y_\kappa,$$

for some permutation $\{\imath, \jmath, \kappa\} = \{1, 2, 3\}$. Our plan is to use Lemma 11 whenever

$$Y_\kappa \leqslant (Y_\imath Y_\jmath)^9 \quad \text{or} \quad U \leqslant (Y_\imath Y_\jmath)^3, \tag{6.24}$$

and Lemma 12 in the alternative case

$$Y_\imath Y_\jmath \leqslant \min\bigl\{Y_\kappa^{1/9}, U^{1/3}\bigr\}. \tag{6.25}$$

Let us consider the case (6.24) first. Recalling the definition (6.19) of σ, it follows that

$$\sigma \ll 1 + (Y_\imath Y_\jmath)^{10\varepsilon - 1/16}.$$

Hence, we may take $\sigma \ll 1$ in Lemma 11, provided that we take $\varepsilon > 0$ to be sufficiently small. It therefore follows from (6.8) and Lemma 11 that

$$\mathcal{N}_1 \ll (X_1X_2X_3)^{1/6}X_4^{1/2} + S_0S^{1/2}UY^{1/2}\left(1 + \frac{\log B}{(Y_\imath Y_\jmath)^{1/16}}\right), \tag{6.26}$$

whenever (6.24) holds. We now sum over the various dyadic intervals for $S_0, S_i, U_i, Y_i, X_i, X_4$ subject to (6.4)–(6.6) and (6.24). Suppose for the moment that we want to sum over all possible dyadic intervals $X \leqslant |x| < 2X$, for which $|x| \leqslant \mathcal{X}$. Then there are plainly $O(\log \mathcal{X})$ possible choices for X. In addition to this basic estimate, we shall make frequent use of the estimates

$$\sum_X X^\delta \ll_\delta \begin{cases} 1, & \delta < 0, \\ \mathcal{X}^\delta, & \delta > 0. \end{cases}$$

Returning to (6.26), we may deduce from (6.5) and (6.7) that values of S_0, Y_1, Y_2, Y_3 are determined by the choices of X_1, X_2, X_3, X_4 and S_i, U_i. Now there are clearly $\ll (\log B)^6$ possible sets of values for S_i, U_i. In view of (6.4), we therefore obtain the estimate

$$\sum_{S_0, S_i, U_i, Y_i, X_i, X_4} (X_1X_2X_3)^{1/6}X_4^{1/2} \ll B(\log B)^6. \tag{6.27}$$

Employing (6.9), we find similarly that

$$\sum_{S_0, S_i, U_i, Y_i, X_i, X_4} S_0S^{1/2}UY^{1/2} \ll \sum_{S_0, S_i, U_i, Y_i, X_i, X_4} (X_1X_2X_3X_4)^{1/4} \ll B(\log B)^6. \tag{6.28}$$

Finally, we turn to the term $S_0S^{1/2}U(Y_\imath Y_\jmath)^{7/16}Y_\kappa^{1/2}\log B$ in (6.26). We shall sum over dyadic intervals subject to the two inequalities

$$Y_\kappa \ll \frac{B}{Y_\imath Y_\jmath}, \qquad S_1 \ll \frac{B}{S_0^3S_2S_3U^2}.$$

The first of these follows from (6.4) and (6.6), whereas the second is just (6.22). We therefore obtain the estimate

$$\begin{aligned}\sum_{S_0,S_i,U_i,Y_i} S_0 S^{1/2} U (Y_\iota Y_J)^{7/16} Y_\kappa^{1/2} &\ll B^{1/2} \sum_{S_0,S_i,U_i,Y_\iota,Y_J} \frac{S_0 S^{1/2} U}{(Y_\iota Y_J)^{1/16}} \\ &\ll B \sum_{S_0,S_2,S_3,U_i,Y_\iota,Y_J} S_0^{-1/2} (Y_\iota Y_J)^{-1/16} \ll B(\log B)^5.\end{aligned}$$

Since values of X_1, X_2, X_3, X_4 are determined by choices of S_0, S_i, U_i, Y_i, we may combine this latter estimate with (6.27) and (6.28) in (6.26), in order to conclude that

$$\sum_{\substack{S_0,S_i,U_i,Y_i,X_i,X_4\\ \text{(6.24) holds}}} \mathcal{N}_1 \ll B(\log B)^6. \tag{6.29}$$

Next we handle the case in which (6.25) holds. For this we employ the alternative estimate Lemma 12 to deduce that

$$\mathcal{N}_1 \ll (S_0 S_\kappa U Y_\iota Y_J + S_0 S_\iota S_J U_\iota U_J Y_\iota Y_J)(\log B)^2.$$

Again we proceed by summing over dyadic intervals for S_0, S_i, U_i, Y_i, X_i, X_4, this time subject to (6.4)–(6.6) and (6.25). Let us consider the term $S_0 S_\kappa U Y_\iota Y_J$. But then (6.5), (6.7), (6.23) and (6.25) together imply that

$$\begin{aligned}S_0 S_\kappa U Y_\iota Y_J &= S_0^3 S U^2 \frac{Y_\iota Y_J}{S_0^2 S_\iota S_J U} \ll (X_1 X_2 X_3)^{1/2} \frac{(Y_\iota Y_J)^{3/2} (U_\iota U_J)^{1/2}}{(X_\iota X_J)^{1/2}} \\ &\ll X_\kappa^{1/2} Y_\kappa^{1/6} (U_\iota U_J)^{1/2} \ll B^{11/12}.\end{aligned}$$

Since there are at most $O_\varepsilon(B^\varepsilon)$ dyadic intervals for S_0, S_i, U_i, Y_i, which in turn determine values of X_1, X_2, X_3, X_4, this therefore leads to the conclusion that

$$\sum_{S_0,S_i,U_i,Y_i,X_i,X_4} S_0 S_\kappa U Y_\iota Y_J (\log B)^2 \ll B, \tag{6.30}$$

whenever (6.25) holds. Lastly we consider the term $S_0 S_\iota S_J U_\iota U_J Y_\iota Y_J$. Now there are $O(\log B)$ dyadic intervals for Y_κ, and (6.25) implies that $Y_\iota, Y_J \leqslant U^{1/3}$. Employing the upper bound $S_\iota \ll B/(S_0^3 S_J S_\kappa U^2)$, we therefore deduce that

$$\begin{aligned}\sum_{S_0,S_i,U_i,Y_i,X_i,X_4} S_0 S_\iota S_J U_\iota U_J Y_\iota Y_J &\ll \log B \sum_{S_0,S_i,U_i} S_0 S_\iota S_J U_\iota U_J U^{2/3} \\ &\ll B \log B \sum_{S_0,S_J,S_\kappa,U_i} S_0^{-2} S_\kappa^{-1} U^{-1/3} \ll B(\log B)^2,\end{aligned}$$

whenever (6.25) holds. Once combined with (6.30) this yields the overall contribution

$$\sum_{\substack{S_0,S_i,U_i,Y_i,X_i,X_4\\ \text{(6.25) holds}}} \mathcal{N}_1 \ll B(\log B)^4. \tag{6.31}$$

Once taken together, (6.29) and (6.31) therefore complete the proof of Proposition 4.

6.2. *Proof of Proposition 5*

We begin this section by providing an upper bound for $\mathcal{N}_2$. For any fixed choice of $s_0 \in \mathbb{N}$ and $\mathbf{s}, \mathbf{u} \in Z_*^3$ in the region (6.2), with (4.9) holding, let $\hat{\mathbf{y}}$ be the vector (6.10) counted by $N(s_0, \mathbf{s}, \mathbf{u})$ that was selected at the start of Section 6. Then (6.12) implies that

$$\mathcal{N}_2 = \sum_{s_0,\mathbf{s},\mathbf{u}} N_2(s_0, \mathbf{s}, \mathbf{u}; \hat{\mathbf{y}}),$$

where $N_2(s_0, \mathbf{s}, \mathbf{u}; \hat{\mathbf{y}})$ denotes the contribution to $N(s_0, \mathbf{s}, \mathbf{u})$ from those $\mathbf{y}$ for which

$$\prod_{1\leqslant i\leqslant 3} (y_i - \hat{y}_i) = 0.$$

Let $N_2^{(i)}$ denote the total contribution to $N_2(s_0, \mathbf{s}, \mathbf{u}; \hat{\mathbf{y}})$ from those $\mathbf{y}$ for which $y_i = \hat{y}_i$ is fixed. It therefore follows that

$$\mathcal{N}_2 \leqslant \sum_{s_0,\mathbf{s},\mathbf{u}} \bigl(N_2^{(1)} + N_2^{(2)} + N_2^{(3)}\bigr) = \mathcal{N}_2^{(1)} + \mathcal{N}_2^{(2)} + \mathcal{N}_2^{(3)}, \tag{6.32}$$

say. In order to estimate $N_2^{(i)}$ for fixed values of $s_0 \in \mathbb{N}$ and $\mathbf{s}, \mathbf{u} \in Z_*^3$, it suffices to count nonzero integer solutions y_j, y_k to the equation

$$y_j u_j s_j^2 + y_k u_k s_k^2 = n, \tag{6.33}$$

where $n = s_0 s_1 s_2 s_3 u_1 u_2 u_3 - \hat{y}_i u_i s_i^2$ is fixed. Our first step is to deduce from (4.9) that

$$\gcd\bigl(u_i s_i^2, u_j s_j^2\bigr) = 1.$$

Noting that $|y_j| < 2Y_j$ and $|y_k| < 2Y_k$, we proceed by applying Lemma 1 to (6.33). Taking

$$\mathbf{h} = \bigl(u_j s_j^2, u_k s_k^2, n\bigr), \qquad \mathbf{w} = (y_j, y_k, 1),$$

we therefore deduce that

$$N_2^{(i)} \ll 1 + \frac{Y_j Y_k}{\max\{Y_j U_j S_j^2, Y_k U_k S_k^2, |n|\}}. \tag{6.34}$$

Since $Y_i U_i S_i^2 \ll \hat{y}_i u_i s_i^2 \ll Y_i U_i S_i^2$ and $S_0 SU \ll s_0 s_1 s_2 s_3 u_1 u_2 u_3 \ll S_0 SU$, by (6.2), it is easy to see that

$$|n| = \bigl|s_0 s_1 s_2 s_3 u_1 u_2 u_3 - \hat{y}_i u_i s_i^2\bigr| \geqslant \bigl|\bigl|\hat{y}_i u_i s_i^2\bigr| - |s_0 s_1 s_2 s_3 u_1 u_2 u_3|\bigr| \gg Y_i U_i S_i^2,$$

if $Y_iU_iS_i^2 \gg S_0SU$. Upon summing (6.34) over all s_0, $\mathbf{s}$, $\mathbf{u}$, and then inserting the resulting bound into (6.32), we therefore obtain the following result:

Lemma 13. *We have*

$$\mathcal{N}_2 \ll S_0SU + \max_{\{i,j,k\}} \left\{ \frac{S_0SUY_jY_k}{\max\{Y_jU_jS_j^2, Y_kU_kS_k^2, \theta_i\}} \right\},$$

where the first maximum is over all permutations $\{i, j, k\}$ *of the set* $\{1, 2, 3\}$*, and*

$$\theta_i = \begin{cases} Y_iU_iS_i^2, & Y_iU_iS_i^2 \gg S_0SU, \\ 1, & \text{otherwise.} \end{cases} \tag{6.35}$$

We now complete the proof of Proposition 5. Recall inequality (6.21). Our first task will be to establish that

$$\sum_{S_0,S_i,U_i,Y_i,X_i,X_4} Y_3U_3S_3^2 \ll B(\log B)^4, \tag{6.36}$$

where the summation is over dyadic intervals subject to (6.4)–(6.6). In order to do so we observe as in Section 6.1 that values of X_1, X_2, X_3, X_4 are determined by the choices of S_0, S_i, U_i, Y_i. Recall the inequalities (6.20). We have two basic cases to consider, according to whether or not $Y_3U_3S_3^2$ is sufficiently large compared with $Y_2U_2S_2^2$.

Suppose first that $Y_3U_3S_3^2 \gg Y_2U_2S_2^2$. Then the ranges (6.2) imply that

$$\left|y_1u_1s_1^2 + y_2u_2s_2^2 + y_3u_3s_3^2\right| \geqslant \left|\left|y_3u_3s_3^2\right| - \left|y_1u_1s_1^2 + y_2u_2s_2^2\right|\right| \gg Y_3U_3S_3^2,$$

in any solution. Since we obviously have $|y_1u_1s_1^2 + y_2u_2s_2^2 + y_3u_3s_3^2| \ll Y_3U_3S_3^2$, the basic equation (4.8) implies that $Y_3U_3S_3^2 \ll S_0SU \ll Y_3U_3S_3^2$, whence

$$S_0S_1S_2U_1U_2 \ll Y_3S_3 \ll S_0S_1S_2U_1U_2. \tag{6.37}$$

Summing over $Y_3 \ll S_0S_1S_2S_3^{-1}U_1U_2$, we therefore obtain

$$\sum_{S_0,S_i,U_i,Y_i,X_i,X_4} Y_3U_3S_3^2 \ll \sum_{S_0,S_i,U_i,Y_1,Y_2} S_0SU,$$

where the last sum is subject to the inequality (6.22). Since there are $\ll (\log B)^2$ choices for Y_1, Y_2, we therefore see that this sum is at most

$$\begin{aligned} &\ll (\log B)^2 \sum_{S_0,S_1,S_2,U_i} S_0S_1S_2U \sum_{S_3 \ll B/(S_0^3S_1S_2U^2)} S_3 \\ &\ll B(\log B)^2 \sum_{S_0,S_1,S_2,U_i} S_0^{-2}U_1^{-1}U_2^{-1}U_3^{-1} \ll B(\log B)^4, \end{aligned}$$

as required for (6.36).

Next, if $Y_2U_2S_2^2 \ll Y_3U_3S_3^2 \ll Y_2U_2S_2^2$, then it follows that any choice of Y_3, U_2, U_3, S_2, S_3 determines a choice of Y_2. Proceeding in a similar fashion to above, we deduce from (6.4) and (6.5) that $Y_3 \ll B/(U_3US_0^2S_3^2)$. Hence we obtain the estimate

$$\sum_{S_0,S_i,U_i,Y_i,X_i,X_4} Y_3U_3S_3^2 \ll B \sum_{S_0,S_i,U_i,Y_1} S_0^{-2}U_1^{-1}U_2^{-1}U_3^{-1} \ll B(\log B)^4.$$

This completes the proof of (6.36).

Recall the estimate in Lemma 13 for $\mathcal{N}_2$. Then in view of (6.36), it suffices to estimate

$$N^{(i)}(B) = \sum_{S_0,S_i,U_i,Y_i} \frac{S_0SUY_jY_k}{\max\{Y_jU_jS_j^2, Y_kU_kS_k^2, \theta_i\}}, \tag{6.38}$$

for each permutation $\{i, j, k\}$ of $\{1, 2, 3\}$, and where θ_i is given by (6.35). We begin by handling the case $i = 3$. Suppose first that $Y_3U_3S_3^2 \gg Y_2U_2S_2^2$, so that (6.37) holds and we may take $\theta_3 = Y_3U_3S_3^2$. Then

$$\frac{S_0SUY_1Y_2}{\max\{Y_1U_1S_1^2, Y_2U_2S_2^2, \theta_3\}} = \frac{S_0S_1S_2U_1U_2Y_1Y_2}{S_3Y_3}.$$

Moreover, we recall the inequalities

$$Y_2 \ll \frac{B}{U_2US_0^2S_2^2}, \qquad Y_1 \ll \frac{Y_3U_3S_3^2}{U_1S_1^2}, \qquad S_3 \ll \frac{S_0S_1S_2U_1U_2}{Y_3},$$

which follow from (6.5), (6.20) and (6.37), respectively. But then it follows that

$$N^{(3)}(B) \ll B \sum_{S_0,S_i,U_i,Y_1,Y_3} \frac{S_1Y_1}{S_0S_2S_3U_2U_3Y_3} \ll B \sum_{S_0,S_i,U_i,Y_3} \frac{S_3}{S_0S_1S_2U_1U_2} \ll B \sum_{S_0,S_1,S_2,U_i,Y_3} Y_3^{-1},$$

whence $N^{(3)}(B) \ll B(\log B)^6$ if $Y_3U_3S_3^2 \gg Y_2U_2S_2^2$. Next we suppose that $Y_3U_3S_3^2 \ll Y_2U_2S_2^2$, and take $\theta_3 = 1$ in (6.38). Observe that

$$S_0 \ll \frac{Y_3S_3}{S_1S_2U_1U_2}, \qquad U_3 \ll \frac{Y_2U_2S_2^2}{Y_3S_3^2}, \qquad Y_1 \ll \frac{B}{Y_2Y_3},$$

which follow from (6.21), the inequality $Y_3U_3S_3^2 \ll Y_2U_2S_2^2$ and (6.6), respectively. We then argue as above to deduce that

$$\begin{aligned} N^{(3)}(B) &= \sum_{S_0,S_i,U_i,Y_i} S_0S_1S_2^{-1}S_3U_1U_3Y_1 \ll \sum_{S_i,U_i,Y_i} \frac{Y_1Y_3U_3S_3^2}{U_2S_2^2} \\ &\ll \sum_{S_i,U_1,U_2,Y_i} Y_1Y_2 \ll B \sum_{S_i,U_1,U_2,Y_2,Y_3} Y_3^{-1}. \end{aligned}$$

Hence $N^{(3)}(B) \ll B(\log B)^6$ in this case also.

Finally, we must estimate (6.38), whenever $i \neq 3$. Suppose that $i = 1$, so that

$$N^{(1)}(B) = \sum_{S_0, S_i, U_i, Y_i} \frac{S_0 S U Y_2 Y_3}{Y_3 U_3 S_3^2} = \sum_{S_0, S_i, U_i, Y_i} S_0 S_1 S_2 S_3^{-1} U_1 U_2 Y_2.$$

Once again we separate our arguments according to the size of $Y_3 U_3 S_3^2$. Suppose that $Y_3 U_3 S_3^2 \gg Y_2 U_2 S_2^2$, so that (6.37) holds. Then Y_3 is fixed by the choices of S_0, S_i, U_1, U_2. Moreover, we have the inequalities

$$Y_2 \ll \frac{B}{U_2 U S_0^2 S_2^2}, \qquad S_1 \ll \frac{S_0 S_2 S_3 U_2 U_3}{Y_1},$$

which follow from (6.5) and the fact that $Y_1 U_1 S_1^2 \ll Y_3 U_3 S_3^2 \ll S_0 S U$, respectively. Summing over Y_2, and then over S_1, we therefore deduce that

$$N^{(1)}(B) \ll B \sum_{S_0, S_i, U_i, Y_1} \frac{S_1}{S_0 S_2 S_3 U_2 U_3} \ll B \sum_{S_0, S_2, S_3, U_i, Y_1} Y_1^{-1} \ll B(\log B)^6,$$

in this case.

Alternatively, if $Y_3 U_3 S_3^2 \ll Y_2 U_2 S_2^2$, then U_3 is determined by choices of S_2, S_3, U_2, Y_2, Y_3, and it follows that

$$N^{(1)}(B) \ll \sum_{S_0, S_i, U_1, U_2, Y_i} S_0 S_1 S_2 S_3^{-1} U_1 U_2 Y_2.$$

Upon summing over $S_0 \ll Y_3 S_3/(S_1 S_2 U_1 U_2)$, and then over $Y_2 \ll B/(Y_1 Y_3)$, we derive the estimate

$$N^{(1)}(B) \ll B \sum_{S_i, U_1, U_2, Y_1, Y_3} Y_1^{-1} \ll B(\log B)^6,$$

in this case.

An entirely similar argument handles the case $i = 2$. Upon combining our various estimates we therefore deduce the statement of Proposition 5.

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